KYUNGPOOK Math. J. 60(2020), 387-399
https://doi.org/10.5666/KMJ.2020.60.2.387
pISSN 1225-6951 eISSN 0454-8124

Set shared by an Entire Function with its k-th Derivative Using Normal Families

Molla Basir Ahamed
Department of Mathematics, Kalipada Ghosh Tarai Mahavidyalaya, Bagdogra, West Bengal, 734014, India
e-mail : bsrhmd117@gmail.com

Dedicated to the memory of my beloved mother-in-law, Rehana Mondal (September 16, 1966 November 11, 2016)

Abstract. In this paper, we study a problem of a non-constant entire function f that shares a set $\mathcal{S} = \{a, b, c\}$ with its k-th derivative $f^{(k)}$, where a, b and c are any three distinct complex numbers. We have found a gap in the statement of the main result of *Chang-Fang-Zalcman* [10], and with the help of their method, we have generalize their result in a more compact form. As an application, we generalize the famous Brück conjecture [9] with the idea of set sharing.

1. Introduction, Definitions and Results

It is well known that Nevanlinna theory plays an important role in considering the value distribution of meromorphic functions and non-trivial solutions of some complex differential equations. A function f is called meromorphic if it is analytic in the complex plane $\mathbb{C}$ except at isolated poles. In what follows, we assume that the reader is familiar with the basic Nevanlinna Theory [18, 26]. It will be convenient to let E denote any set of positive real real numbers of finite linear measure, not necessarily the same at each occurrence. Let f and g be two meromorphic functions having the same set of a-points with the same multiplicities, we then say that f and g share the value a CM (counting multiplicities) and if we do not consider the multiplicities then f and g are said to share the value a IM.

When $a = \infty$, the zeros of $f - a$ means the poles of f.

Definition 1.1. For a non-constant meromorphic function f and any set $\mathcal{S} \subset \mathbb{C} \cup \{\infty\}$, we define

Received April 7, 2018; revised June 27, 2019; accepted November 28, 2019.
2010 Mathematics Subject Classification: 30D35.
Key words and phrases: normal families, entire functions, set sharing, derivative.

$$E_f(\mathcal{S}) = \bigcup_{a \in \mathcal{S}} \Big\{(z,p) \in \mathbb{C} \times \mathbb{N} : f(z) = a, \text{ with multiplicity } p\Big\},$$
$$\overline{E}_f(\mathcal{S}) = \bigcup_{a \in \mathcal{S}} \Big\{(z,1) \in \mathbb{C} \times \{1\} : f(z) = a\Big\}.$$

If $E_f(\mathcal{S}) = E_g(\mathcal{S})$ (resp. $\overline{E}_f(\mathcal{S}) = \overline{E}_g(\mathcal{S})$) then we simply say f and g share the set $\mathcal{S}$ Counting Multiplicities (CM) (resp. Ignoring Multiplicities (IM)).

If $\mathcal{S}$ contains one element only, then it coincides with the usual definition of $CM(IM)$ sharing of values.

In 1926, Nevanlinna first showed that a non-constant meromorphic function on the complex plane $\mathbb{C}$ is uniquely determined by the pre-images, ignoring multiplicities, of five distinct values (including infinity). A few years latter, he showed that when multiplicities are taken into consideration, four points are enough and in that case either the two functions coincide or one is the bilinear transformation of the other one.

Recall that the spherical derivative of a meromorphic function f on a plane domain is

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}.$$

The sharing value problem between an entire function and its derivatives was first studied by *Rubel-Yang* [25]. They proved that if a non-constant entire function f and its derivative f' share two distinct finite numbers a, b CM, then $f \equiv f'$.

In 1979, *Mues-Steinmetz* [23] improved the above theorem in the following manner.

Theorem A.([23]) *Let f be a non-constant entire function. If f and f' share two distinct values a, b IM then $f \equiv f'$.*

Remark 1.1. Following example shows that the results of *Rubel-Yang* and *Mues-Steinmetz* are not in general true when value sharing is replaced by set sharing.

Example 1.1. Let $\mathcal{S} = \left\{\frac{a}{3}, \frac{2a}{3}\right\}$, where $a(\neq 0)$ be any complex number. Let $f(z) = e^{-z} + a$, then $E_f(\mathcal{S}) = E_{f'}(\mathcal{S})$ but $f \not\equiv f'$.

So for the uniqueness of an entire function and its derivative sharing a set, the cardinality of the range set should be at least three.

In this regard in 2003, using the properties of normal families, *Fang-Zalcman* [14] obtained the following result.

Theorem B.([14]) *Let $\mathcal{S} = \{0, a, b\}$, where a, b are two non-zero distinct complex numbers satisfying $a^2 \neq b^2$, $a \neq 2b$, $a^2 - ab + b^2 \neq 0$. If for a non-constant entire function f, $E_f(\mathcal{S}) = E_{f'}(\mathcal{S})$, then $f \equiv f'$.*

In order to generalize the range set in the above theorem, in 2007 *Chang-Fang-Zalcman* [10] obtained the following result.

Theorem C.([10]) *Let f be a non-constant entire function and let $\mathcal{S} = \{a, b, c\}$, where a, b and c are distinct complex numbers. If $E_f(\mathcal{S}) = E_{f'}(\mathcal{S})$, then either*

(1) $f(z) = \mathcal{C}e^z$; *or*

(2) $f(z) = \mathcal{C}e^{-z} + \frac{2}{3}(a + b + c)$ *and* $(2a - b - c)(2b - c - a)(2c - a - b) = 0$; *or*

(3) $f(z) = \mathcal{C}e^{\frac{-1 \pm i\sqrt{3}z}{2}} + \dfrac{3 \pm i\sqrt{3}}{6}(a + b + c)$ *and* $a^2 + b^2 + c^2 - ab - bc - ca = 0$,

where $\mathcal{C}$ is a non-zero constant.

Reamrk 1.2. We see from the next example that the conclusion of *Theorem C* need not hold if the CM shared set $\mathcal{S}$ is replaced by an IM shared set.

Example 1.2.([10]) Let $\mathcal{S} = \{-1, 0, 1\}$ and $f(z) = \sin z$ or $\cos z$. Clearly $\overline{E}_f(\mathcal{S}) = \overline{E}_{f'}(\mathcal{S})$ and f takes none of the forms (1) – (3) in Theorem C.

Remark 1.3. In Example 1.2, one may consider k-th derivative of f instead of first derivative, when k is any odd positive integer to get the same conclusion.

Remark 1.4. We have found a little gap in the statement of *Theorem C* as follows.

(i) In the statement of the *Theorem C*, the author should require that *"f be a non-constant entire function having zeros of multiplicities ≥ 1"*.

(ii) It is affirmed that f has zeros, so it is natural to see that the possible form of the function should not be of the form $f(z) = \mathcal{C}e^z$ in *Theorem C* as f has no zero in this particular form.

From the above discussions, one may note that a non-constant entire function f sharing an arbitrary set of three finite complex numbers a, b and c (CM) with its first derivative, must have some specific form.

It is natural to ask the following question:

Question 1.1. Is it possible to extend Theorem C to k-th derivative of f ?

If the answer of *Question 1.1* is found to be affirmative, then it will be interesting to investigate on the following question:

Question 1.2. What can we say about the possible forms of the function f ?

Since f and $f^{(k)}$ share the set $\mathcal{S} = \{a, b, c\}$, one may observe that among all the possible relationship between f and $f^{(k)}$, clearly $f^{(k)} \equiv f$ is the most obvious one. So before going to state our main results, we want to discuss a natural question: *What is the general solution of $f^{(k)} \equiv f$?* The natural answer is $f(z) = \mathcal{L}_\theta(z)$ (see [1, 6]) where we define $\mathcal{L}_\theta(z)$ as follows

$$\mathcal{L}_\theta(z) = c_0e^z + c_1e^{\theta z} + c_2e^{\theta^2 z} + \ldots + c_{k-1}e^{\theta^{k-1} z}, \tag{1.1}$$

where $c_i \in \mathbb{C}$ for $i \in \{0, 1, 2, \ldots, k-1\}$ with $c_{k-1} \neq 0$ and $\theta = \cos\left(\frac{2\pi}{k}\right) + i \sin\left(\frac{2\pi}{k}\right)$.

To answer all the above questions affirmatively is the main motivation of writing this paper. We have tried to take care of the points mentioned in *Remark 1.2*. Following is the main result of this paper.

Theorem 1.1. *Let f be a non-constant entire function, having zeros of multiplicity at least k and let $\mathcal{S} = \{a, b, c\}$, where a, b and c are distinct complex numbers. If $E_f(\mathcal{S}) = E_{f^{(k)}}(\mathcal{S})$, then f takes one of the following forms:*

(1) *$f(z) = \mathcal{L}_\theta(\beta z)$, where β is a root of the equation $z^k - 1 = 0$,*

(2) *$f(z) = \mathcal{L}_\theta(\eta z) + \frac{2}{3}(a + b + c)$, where η is a root of the equation $z^k + 1 = 0$ and $(2a - b - c)(2b - c - a)(2c - a - b) = 0$,*

(3) *$f(z) = \mathcal{L}_\theta(\zeta z) + \frac{3 \pm i\sqrt{3}}{6}(a + b + c)$, where $\zeta (\neq 1)$ is a root of the equation $z^{3k} - 1 = 0$ and $a^2 + b^2 + c^2 - ab - bc - ca = 0$,*

where $\mathcal{L}_\theta(z)$ is defined in (1.1).

Remark 1.5. The conclusion (2) of Theorem 1.1 can not be omitted, following example ensures this fact.

Example 1.3. Let $k = 2$, hence $\theta = -1$. So we have $\eta = \pm i$, and hence $\mathcal{L}_\theta(\eta z) = c_0 e^{\eta z} + c_1 e^{\theta \eta z} = \frac{1}{4}\left(e^{iz} + e^{-iz}\right)$,where $c_0 = \frac{1}{4} = c_1$. Let $\mathcal{S} = \left\{a, \frac{1}{4}, \frac{1}{2} - a\right\}$, where $a \in \mathbb{C} \setminus \{1/4\}$. Let $f(z) = \frac{1}{2}\cos z + \frac{1}{2} = \cos^2\left(\frac{z}{2}\right)$, clearly it is of the form $f(z) = \mathcal{L}_\theta(\eta z) + \frac{2}{3}(a + b + c)$. One can verify that $E_f(\mathcal{S}) = E_{f''}(\mathcal{S})$, and the multiplicities of zeros of $f(z)$ are at least 2.

2. Some Lemmas

We begin our investigation with the following lemmas, which are essential to prove our main results.

Lemma 2.1.([13]) *The order of an entire function having bounded spherical derivative on $\mathcal{C}$ is at most* 1.

Lemma 2.2.([14]) *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D. Let k be a positive integer. Let a, b and c be three distinct finite complex numbers and M a positive number. If, for any $f \in \mathcal{F}$, the zeros of f are of multiplicity $\geq k$ and $|f^{(k)}(z)| \leq M$ whenever $f(z) \in \{a, b, c\}$, then $\mathcal{F}$ is normal in D.*

Lemma 2.3.([15]) *Let f be a non-constant meromorphic function of finite order ρ, and $\epsilon > 0$ a constant. Then there exists a set $E \subset [0, 2\pi)$ which has linear measure* *zero, such that if $\psi_0 \in [0, 2\pi) - E$, then there is a constant $R_0 = R_0(\psi_0) > 0$ such that for all z satisfying $arg z = \psi_0$ and $|z| > R_0$, we have*

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \leq |z|^{k(\rho+\epsilon-1)}.$$

Lemma 2.4. *Let f be an entire function, and suppose that $|f^{(k)}(z)|$ is unbounded on some ray $\arg z = \theta$. Then there exists an infinite sequence of points $z_n = r_n e^{\theta}$ where $r_n \to \infty$, such that $f^{(k)}(z_n) \to \infty$ and*

$$\left| \frac{f(z_n)}{f^{(k)}(z_n)} \right| \leq (1 + o(1))|z_n|^k \tag{2.1}$$

as $z_n \to \infty$.

Proof. For $k \geq 1$, one must have

$$f(z_n) = \sum_{i=0}^{k-1} \frac{z_n^i}{(i)!} f^{(i)}(0) + \left\{ \int_0^{z_n} \overbrace{\int_0^z \cdots \int_0^z}^{(k-1)-\text{times}} f^{(k)}(z) \overbrace{dz \ldots dz}^{k-\text{times}} \right\}. \tag{2.2}$$

By applying the triangle inequality in (2.2), rest of the proof follows from the proof of [16, Lemma 4, page. 421]. □

Lemma 2.5.([22]) *A class $\mathcal{C}$ of functions f meromorphic in a domain $D \subset \mathbb{C}$ is normal in D if and only if $f^{\#}$ is uniformly bounded on any compact subset of D for $f \in \mathcal{C}$.*

Lemma 2.6.([20, 24]) *Let f be an entire function of order at most 1 and k be a positive integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = o(\log r), \quad as\, r \to \infty.$$

Lemma 2.7. *Let α be a non-constant entire function and a, b and c are three distinct finite complex numbers. Then there does not exist an entire function f having zeros of multiplicity $\geq k$, satisfying the differential equation*

$$\frac{\left(f^{(k)} - a\right)\left(f^{(k)} - b\right)\left(f^{(k)} - c\right)}{(f - a)(f - b)(f - c)} = e^{\alpha}. \tag{2.3}$$

Proof. Assume there exists an entire function f satisfying (2.3). Then we see that $|f^{(i)}(z)| \leq \max\{a, b, c\}$ whenever $f(z) \in \{a, b, c\}$, $i \in \{1, 2, ..., k\}$. As the zeros of f have multiplicity $\geq k$, we have, by Lemma 2.2, that the family $\mathcal{F}_w = \{f_w : w \in \mathbb{C}\}$, where $f_w(z) = f(w + z)$, is normal on the unit disc. Hence by Marty's Theorem, we get that $f^{\#}(w) = (f_w)^{\#}(0)$ is uniformly bounded for all $w \in \mathbb{C}$. Therefore in view of Lemma 2.1, we get that f has order at most 1.

From (2.3), we obtain $\alpha(z) = \mathcal{A}z + \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are two constants. It is clear that $\mathcal{A} \neq 0$, since α is non-constant.

Next we claim that $abc \neq 0$. On contrary, let $abc = 0$. i.e., $a = 0$ or $b = 0$ or $c = 0$. Without any loss of generality, we may assume that $a = 0$. It follows from (2.3) that

$$\frac{f^{(k)}\left(f^{(k)} - b\right)\left(f^{(k)} - c\right)}{f(f-b)(f-c)} = e^{\mathcal{A}z+\mathcal{B}}.$$

Again we see that

$$\begin{aligned}
&\frac{f^{(k)}\left(f^{(k)} - b\right)\left(f^{(k)} - c\right)}{f(f-b)(f-c)} \\
&= \frac{\left(f^{(k)}\right)^3}{f(f-b)(f-c)} - \frac{(b+c)\left(f^{(k)}\right)^2}{f(f-b)(f-c)} + \frac{bcf^{(k)}}{f(f-b)(f-c)} \\
&= \frac{f^{(k)}}{f}\frac{f^{(k)}}{f-b}\frac{f^{(k)}}{f-c} - \frac{b+c}{b-c}\left(\frac{f^{(k)}}{f-b} - \frac{f^{(k)}}{f-c}\right) + bc\left(\frac{\mathcal{A}_1 f^{(k)}}{f} + \frac{\mathcal{B}_1 f^{(k)}}{f-b} + \frac{\mathcal{B}_1 f^{(k)}}{f-c}\right),
\end{aligned}$$

where $\mathcal{A}_1, \mathcal{B}_1$ and $\mathcal{C}_1$ are constants. So there exists $\mathcal{A}_2, \mathcal{B}_2$ and $\mathcal{C}_2$ such that

$$\begin{aligned}
&m\left(r, \frac{f^{(k)}(f^{(k)} - b)(f^{(k)} - c)}{f(f-b)(f-c)}\right) \\
&\leq \mathcal{A}_2\, m\left(r, \frac{f^{(k)}}{f}\right) + \mathcal{B}_2\, m\left(r, \frac{f^{(k)}}{f-b}\right) + \mathcal{C}_2\, m\left(r, \frac{f^{(k)}}{f-c}\right) + O(1).
\end{aligned}$$

Thus by Lemma 2.6, we deduce that

$$T(r, e^{\mathcal{A}z+\mathcal{B}}) = m(r, e^{\mathcal{A}z+\mathcal{B}}) = o(\log r),$$

and this is not possible since $\mathcal{A} \neq 0$.

Therefore, we must have $abc \neq 0$. We set

$$g(z) = f(z/\mathcal{A}) \ \ i.e.,\ \ g^{(k)}(z) = \frac{1}{\mathcal{A}^k} f^{(k)}(z/\mathcal{A}). \tag{2.4}$$

Hence (2.4) and (2.3) mean that

$$\frac{\left(g^{(k)} - a/\mathcal{A}^k\right)\left(g^{(k)} - b/\mathcal{A}^k\right)\left(g^{(k)} - c/\mathcal{A}^k\right)}{(g-a)(g-b)(g-c)} \equiv \mathcal{C}e^z, \tag{2.5}$$

where $\mathcal{C} = \dfrac{e^{\mathcal{B}}}{\mathcal{A}^{3k}} \neq 0$. We see that (2.5) can be written as

$$\frac{\left(g^{(k)}\right)^3 + \mathcal{C}_1\left(g^{(k)}\right)^2 + \mathcal{C}_2 g^{(k)}}{(g-a)(g-b)(g-c)} - \mathcal{C}e^z = \frac{\mathcal{C}_3}{(g-a)(g-b)(g-c)}, \tag{2.6}$$

where $\mathcal{C}_j$ are constants with $\mathcal{C}_3 \neq 0$. With $\epsilon = \dfrac{1}{3}$, Lemma 2.3 shows that there exists a set $E \subset [0, 2\pi)$ of measure zero such that for each $\psi_0 \in [0, 2\pi) - E$, there is a constant $R_0 = R_0(\psi_0) > 0$ such that whenever $\arg z = \psi_0$ and $|z| > R_0$,

$$\left| \frac{\left(g^{(k)}\right)^3 + \mathcal{C}_1 \left(g^{(k)}\right)^2 + \mathcal{C}_2 g^{(k)}}{(g-a)(g-b)(g-c)} \right| \leq K|z|, \tag{2.7}$$

for some positive constant K. We may now suppose that $\pi/2$ and $3\pi/2$ are contained in the set E. Then $[0, 2\pi) - E = E_1 \cup E_2$, where $E_1 = \{\theta \in [0, 2\pi) : \cos\theta > 0\}$ and $E_2 = \{\theta \in [0, 2\pi) : \cos\theta < 0\}$. Let $\theta \in E_1$, then by (2.6) and (2.7), we get for sufficiently large r,

$$\begin{aligned} &\left| \frac{\mathcal{C}_3}{(g(re^{i\theta}) - a)\,(g(re^{i\theta}) - b)\,(g(re^{i\theta}) - c)} \right| \\ &= \left| \frac{\left(g^{(k)}(re^{i\theta})\right)^3 + \mathcal{C}_1 \left(g^{(k)}(re^{i\theta})\right)^2 + \mathcal{C}_2 g^{(k)}(re^{i\theta})}{(g(re^{i\theta}) - a)\,(g(re^{i\theta}) - b)\,(g(re^{i\theta}) - c)} - \mathcal{C}e^{re^{i\theta}} \right| \\ &\geq |\mathcal{C}|e^{r\cos\theta} - Kr \\ &\to \infty, \quad \text{as} \quad r \to \infty. \end{aligned}$$

It follows that

$$g(re^{i\theta}) \to a,\ b \ \text{ or } \ c, \quad \text{as} \quad r \to \infty. \tag{2.8}$$

Let $\theta \in E_2$. We claim that $|g^{(k)}(re^{i\theta})|$ is bounded as $r \to \infty$. On contrary, we suppose that $|g^{(k)}(re^{i\theta})|$ is unbounded as $r \to \infty$. Then by Lemma 2.4, there exists a sequence $r_n \to \infty$ such that $|g^{(k)}(re^{i\theta})| \to \infty$ and

$$\left| g(re^{i\theta}) g^{(k)}(re^{i\theta}) \right| \leq (1 + o(1)) r_n^k. \tag{2.9}$$

With $|g^{(k)}(r_n e^{i\theta})| \to \infty$, we note that

$$\left| \frac{\left(g(r_n e^{i\theta}) - a\right)\left(g(r_n e^{i\theta}) - b\right)\left(g(r_n e^{i\theta}) - c\right)}{\left(g^{(k)}(r_n e^{i\theta})\right)^3 + \mathcal{C}_1 \left(g^{(k)}(r_n e^{i\theta})\right)^2 + \mathcal{C}_2 g^{(k)}(r_n e^{i\theta})} \right| \leq (1 + o(1)) r_n^{3k}. \tag{2.10}$$

Since $|g^{(k)}(r_n e^{i\theta})| \to \infty$, so it follows from (2.6) that

$$\begin{aligned} &\left| \frac{\left(g(r_n e^{i\theta}) - a\right)\left(g(r_n e^{i\theta}) - b\right)\left(g(r_n e^{i\theta}) - c\right)}{r_n^{3k}\mathcal{C}_3} \right| \\ &= \left| \frac{\left(g^{(k)}(r_n e^{i\theta})\right)^3 + \mathcal{C}_1 \left(g^{(k)}(r_n e^{i\theta})\right)^2 + \mathcal{C}_2 g^{(k)}(r_n e^{i\theta}) - \mathcal{C}_3}{r_n^{3k}|\mathcal{C}_3||\mathcal{C}|e^{r_n e^{i\theta}}} \right| \\ &= r_n^{-3k} e^{-r_n\cos\theta} \frac{\left| \left(g^{(k)}(r_n e^{i\theta})\right)^3 + \mathcal{C}_1 \left(g^{(k)}(r_n e^{i\theta})\right)^2 + \mathcal{C}_2 g^{(k)}(r_n e^{i\theta}) - \mathcal{C}_3 \right|}{|\mathcal{C}_3\mathcal{C}|} \\ &\to \infty. \end{aligned} \tag{2.11}$$

Thus from (2.6), (2.10) and (2.11), we get

$$\begin{aligned}
&1-o(1)\\
&\leq \left| r_n^{3k} \frac{\left(g^{(k)}(r_ne^{i\theta})\right)^3+\mathcal{C}_1\left(g^{(k)}(r_ne^{i\theta})\right)^2+\mathcal{C}_2 g^{(k)}(r_ne^{i\theta})-\mathcal{C}_3}{\left(g(r_ne^{i\theta})-a\right)\left(g(r_ne^{i\theta})-b\right)\left(g(r_ne^{i\theta})-c\right)}\right|\\
&\leq \left|\frac{r_n^{3k}\mathcal{C}_3}{\left(g(r_ne^{i\theta})-a\right)\left(g(r_ne^{i\theta})-b\right)\left(g(r_ne^{i\theta})-c\right)}\right| + |\mathcal{C}| r_n^{3k} e^{r_n\cos\theta}\\
&\to\ 0,
\end{aligned}$$

which is absurd. Hence $|g^{(k)}(r_ne^{i\theta})|$ is bounded as $r\to\infty$ for each $\theta\in E_2$. For a positive integer k, we can always write

$$g(re^{i\theta}) = \sum_{i=0}^{k-1}\frac{z_n^i}{(i)!}f^{(i)}(0) + \left(e^{i\theta}\right)^k\left\{\int_0^r \overbrace{\int_0^t\cdots\int_0^t}^{(k-1)-\text{times}} f^{(k)}(z)\,\overbrace{dt\ldots dt}^{k-\text{times}}\right\}.$$

So we must have

$$\begin{aligned}
|g(re^{i\theta})| &\leq \left|\sum_{i=0}^{k-1}\frac{z_n^i}{(i)!}f^{(i)}(0)\right| + \left|\left\{\int_0^r \overbrace{\int_0^t\cdots\int_0^t}^{(k-1)-\text{times}} f^{(k)}(z)\,\overbrace{dt\ldots dt}^{k-\text{times}}\right\}\right|\\
&\leq \left|\sum_{i=0}^{k-1}\frac{z_n^i}{(i)!}f^{(i)}(0)\right| + \mathcal{M}r^k,
\end{aligned}\tag{2.12}$$

where $\mathcal{M}=\mathcal{M}(\theta)$ is a positive constant depending on θ.

Hence by (2.8) and (2.12), for every $\theta\in[0,2\pi)-E$, there exists a positive constant $\mathcal{L}=\mathcal{L}(\theta)$ such that for $z=re^{i\theta}$ with $r>r_0$,

$$\left|\frac{g(z)}{z^k}\right|\leq\mathcal{L}.\tag{2.13}$$

Since the order of g is at most 1, it follows from (2.8), (2.13), the Phragěn-Lindelöf Theorem, and Liouville's Theorem, that g is a polynomial of degree at most k, which is impossible by (2.5). This completes the proof. $\square$

Lemma 2.8. *Let $\mathcal{S}=\{a,b,c\}$ where a, b and c be any three distinct finite complex numbers and $\mathcal{A}$ a non-zero constant. If $E_f(S)=E_{f^{(k)}}(S)$, where f is an entire function having zeros of multiplicities $\geq k$ and satisfying $f^{(k)}\not\equiv 0$ and*

$$\frac{\left(f^{(k)}-a\right)\left(f^{(k)}-b\right)\left(f^{(k)}-c\right)}{(f-a)(f-b)(f-c)}\equiv\mathcal{A},\tag{2.14}$$

then f must take one of the following forms:

(1) $f(z) = \mathcal{L}_\theta(\beta z)$, *where* β *is a root of the equation* $z^k - 1 = 0$,

(2) $f(z) = \mathcal{L}_\theta(\eta z) + \dfrac{2}{3}(a+b+c)$, *where* η *is a root of the equation* $z^k + 1 = 0$ *and* $(2a-b-c)(2b-c-a)(2c-a-b) = 0$,

(3) $f(z) = \mathcal{L}_\theta(\zeta z) + \frac{3\pm i\sqrt{3}}{6}(a+b+c)$, *where* $\zeta(\neq 1)$ *is a root of the equation* $z^{3k} - 1 = 0$ *and* $a^2+b^2+c^2-ab-bc-ca = 0$,

where $\mathcal{C}$ *is a non-zero constant.*

Proof. From the proof of Lemma 2.7, we note that f has order at most 1. Since f and $f^{(k)}$ have the same order and f having zeros of multiplicities $\geq k$ satisfying $f^{(k)} \not\equiv 0$ and also $E_f(\mathcal{S}) = E_{f^{(k)}}(\mathcal{S})$, so one can deduce that the form is

$$f^{(k)}(z) = c_0\alpha^k e^{\alpha z} + c_1\alpha^k e^{\alpha\theta z} + \ldots + c_{k-1}\alpha^k e^{\alpha\theta^{k-1}z} = \alpha^k\mathcal{L}_\theta(\alpha z), (\text{say}) \tag{2.15}$$

where $c_i \in \mathbb{C}$, for $i \in \{0,1,2,\ldots,k-1\}$ with $c_{k-1} \neq 0$, $\alpha \in \mathbb{C} - \{0\}$,

$$\theta = \cos\left(\frac{2\pi}{k}\right) + i\sin\left(\frac{2\pi}{k}\right)$$

and

$$\mathcal{L}_\theta(\alpha z) = c_0 e^{\alpha z} + c_1 e^{\alpha\theta z} + \ldots + c_{k-1}e^{\alpha\theta^{k-1}z}.$$

Integrating (2.15) k-times, we get

$$f(z) = \mathcal{L}_\theta(\alpha z) + \mathcal{Q}_{k-1}(z), \tag{2.16}$$

where $\mathcal{Q}_{k-1}$ is a polynomial of degree $\leq k-1$.

Using (2.15) and (2.16), we get from (2.14)

$$\begin{aligned} &(\alpha^{3k} - \mathcal{A})\left(\mathcal{L}_\theta(\alpha z)\right)^3 + \left(\mathcal{L}_1\alpha^{2k} - 3\mathcal{A}\mathcal{Q}_{k-1} - \mathcal{A}\mathcal{L}_1\right)\left(\mathcal{L}_\theta(\alpha z)\right)^2 \\ &+ \left(\mathcal{L}_2\alpha^k - 3\mathcal{A}\mathcal{Q}_{k-1}^2 - 2\mathcal{A}\mathcal{L}_1\mathcal{Q}_{k-1} - \mathcal{A}\mathcal{L}_2\right)\mathcal{L}_\theta(\alpha z) \\ &+ \left(\mathcal{L}_3 - \mathcal{A}\mathcal{Q}_{k-1}^3 - \mathcal{A}\mathcal{L}_1\mathcal{Q}_{k-1}^2 - \mathcal{A}\mathcal{L}_2\mathcal{Q}_{k-1} - \mathcal{A}\mathcal{L}_3\right) \equiv 0, \end{aligned} \tag{2.17}$$

where $\mathcal{L}_1 = -(a+b+c)$, $\mathcal{L}_2 = ab+bc+ca$ and $\mathcal{L}_3 = -abc$. It follows that

$$\alpha^{3k} = \mathcal{A}, \tag{2.18}$$

$$\mathcal{L}_1\alpha^{2k} = \mathcal{A}(3\mathcal{Q}_{k-1} + \mathcal{L}_1), \tag{2.19}$$

$$\mathcal{L}_2\alpha^k = \mathcal{A}(3\mathcal{Q}_{k-1}^2 + 2\mathcal{L}_1\mathcal{Q}_{k-1} + \mathcal{L}_2), \tag{2.20}$$

$$\mathcal{L}_3 = \mathcal{A}(\mathcal{Q}_{k-1}^3 + \mathcal{L}_1\mathcal{Q}_{k-1}^2 + \mathcal{L}_2\mathcal{Q}_{k-1} + \mathcal{L}_3). \tag{2.21}$$

We now discuss the following three possible cases.

Case 1. Let $\alpha \in \{z : z^k - 1 = 0\}$. Then from (2.18) and (2.19), we get $\mathcal{A} = 1$ and $\mathcal{Q}_{k-1} = 0$. Thus we see that

$$f(z) = \mathcal{L}_\theta(\beta z),$$

where β is a root of the equation $z^k - 1 = 0$.
Case 2. Let $\alpha \in \{z : z^k + 1 = 0\}$. Then from (2.18) and (2.19), we see that $\mathcal{A} = -1$ and $\mathcal{Q}_{k-1} = -\frac{2}{3}\mathcal{L}_1$. It follows from (2.21) that

$$2\mathcal{L}_1^3 - 9\mathcal{L}_1\mathcal{L}_2 + 27\mathcal{L}_3 = 0$$

which in turn implies that

$$(2a - b - c)(2b - c - a)(2c - a - b) = 0.$$

In this case, we get

$$f(z) = \mathcal{L}_\theta(\eta z) + \frac{2}{3}(a + b + c),$$

where η is a root of the equation $z^k + 1 = 0$.
Case 3. Let $\alpha \notin \{z : z^k - 1 = 0\} \cup \{z : z^k + 1 = 0\}$. Then by (2.18) and (2.19), we get

$$\mathcal{Q}_{k-1} = \frac{1 - \alpha^k}{3\alpha^k}\mathcal{L}_1. \tag{2.22}$$

Then by (2.18), (2.20) and (2.22), we obtained

$$\mathcal{L}_2 = \frac{(\mathcal{L}_1)^2}{3}. \tag{2.23}$$

Next by (2.18), (2.21), (2.22) and (2.23), we also get

$$(1 - \alpha^{3k})\mathcal{L}_3 = \frac{1}{27}(1 - \alpha^{3k})\mathcal{L}_1^3. \tag{2.24}$$

Subcase 3.1. If $\alpha^{3k} \neq 1$, then $\mathcal{L}_3 = (\mathcal{L}_1)^3/27$. This with (2.23) shows that $a = b = c$, which is not possible.

Subcase 3.2. Hence $\alpha^{3k} - 1 = 0$. i.e., $\alpha^k = \dfrac{-1 \pm i\sqrt{3}}{2}$. Thus we have $\mathcal{Q}_{k-1} = -\dfrac{3 \pm i\sqrt{3}}{6}\mathcal{L}_1$. Simplifying (2.23), we get $a^2 + b^2 + c^2 - ab - bc - ca = 0$.

Thus we have

$$f(z) = \mathcal{L}_\theta(\zeta z) + \frac{3 \pm i\sqrt{3}}{6}(a + b + c),$$

where $\zeta(\neq 1)$ is a root of the equation $z^{3k} - 1 = 0$. □

3. Proof of Theorem 1.1

Since $E_f(\mathcal{S}) = E_{f^{(k)}}(\mathcal{S})$, therefore it is clear that

$$\frac{\left(f^{(k)} - a\right)\left(f^{(k)} - b\right)\left(f^{(k)} - c\right)}{(f-a)(f-b)(f-c)} \equiv e^{\alpha(z)}, \tag{3.1}$$

where α is an entire function. We note that by *Lemma 2.7*, α is a constant. Then we set $\mathcal{A} = e^{\alpha}$. Thus (3.1) changes to

$$\frac{\left(f^{(k)} - a\right)\left(f^{(k)} - b\right)\left(f^{(k)} - c\right)}{(f-a)(f-b)(f-c)} \equiv \mathcal{A}. \tag{3.2}$$

Next we discuss the following two cases.
Case 1. If $f^{(k)} \neq 0$, then by *Lemma 2.8*, we see that f takes one of the three forms (1)-(3). So we are done.
Case 2. If $f^{(k)}$ vanishes at some point $z_0 \in \mathbb{C}$. i.e., $f^{(k)}(z_0) = 0$.
Differentiating both sides of (3.2), we get

$$\begin{aligned} &\left\{3\left(f^{(k)}\right)^2 - 2(a+b+c)f^{(k)} + (ab+bc+ca)\right\}f^{(k+1)} \\ &\equiv \mathcal{A}\left\{3f^2 - 2(a+b+c)f + (ab+bc+ca)\right\}f'. \end{aligned} \tag{3.3}$$

Let $f^{(k)}(z_0) = 0$ and $k \leq n$. So we may assume

$$f(z) = f(z_0) + A_n(z-z_0)^n + \ldots$$

Clearly we have $f^{(k)}(z) = B_n(z-z_0)^{n-k} + \ldots$ and $f'(z) = nA(z-z_0)^{n-1} + \ldots$. We see that *L.H.S* of (3.3) vanishes at z_0 with order $n-k$ while *R.H.S* of (3.3) vanishes with the order at least $n-1$, which is not possible.

4. Some Applications

In 1996, the following conjecture was proposed by Brück [9].

Conjecture 4.1.([9]) *Let f be a non-constant entire function. Suppose that $\rho_1(f)$ is not a positive integer or infinite. If f and f' share one finite value a CM, then*

$$\frac{f'-a}{f-a} = c,$$

for some non-zero constant c, where $\rho_1(f)$ is the first iterated order of f which is defined by

$$\rho_1(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

Many authors (for the case of differences see [19, 21] and for the cases of derivatives or differential polynomials see [2, 3, 4, 5, 6, 7, 8] and [11, 12, 17]) have studied the conjecture under some additional conditions. But the main conjecture is still open. In this direction, it is interesting to ask the following two questions.

Question 4.1. Does the conjecture hold if one considers a set having three arbitrary finite complex numbers instead of a value ?

Question 4.2. Is it possible to replace first derivative f' by a more general derivative $f^{(k)}$?

Remark 4.1. Note that *Lemma 2.8* answers the above questions in some sense.

Acknowledgements. The author would like to thank the referees for their constructive comments that led to better presentation of the paper.

References

[1] M. B. Ahamed, *Uniqueness of two differential polynomials of a meromorphic function sharing a set*, Commun. Korean Math. Soc., **33(4)**(2018), 1181–1203.

[2] M. B. Ahamed and A. Banerjee, *Rational function and differential polynomial of a meromorphic function sharing a small function*, Bull. Transilv. Univ. Barsov Ser. III, **10(59)**(2017), 1–17.

[3] A. H. H. Al-Khaladi, *On meromorphic functions that share one value with their derivative*, Analysis (Munich), **25(2)**(2005), 131-140.

[4] A. Banerjee and M. B. Ahamed, *Meromorphic function sharing a small function with its differential polynomial*, Acta Univ. Palacki. Olomuc. Fac. Rerum Natur. Math., **54(1)**(2015), 33–45.

[5] A. Banerjee and M. B. Ahamed, *Uniqueness of a polynomial and a differential monomial sharing a small function*, An. Univ. Vest Timis. Ser. Math.-Inform., **54(1)**(2016), 55–71.

[6] A. Banerjee and M. B. Ahamed, *Polynomial of a meromorphic function and its k-th derivative sharing a set*, Rend. Circ. Mat. Palermo Ser. II, **67(3)**(2018), 581–598.

[7] A. Banerjee and M. B. Ahamed, *Yu's result - a further extension*, Electron. J. Math. Anal. Appl., **6(2)**(2018), 330–348.

[8] A. Banerjee and M. B. Ahamed, *Further investigations on some results of Yu*, J. Class. Anal., **14(1)**(2019), 1–16.

[9] R. Brück, *On entire functions which share one value CM with their first derivative*, Results Math., **30**(1996), 21–24.

[10] J. Chang, M. Fang and L. Zalcman, *Entire functions that share a set with their derivatives*, Arch. Math., **89**(2007), 561–569.

[11] Z. X. Chen and K. H. Shon, *On conjecture of R. Brück concerning the entire function sharing one value CM with its derivative*, Taiwanese J. Math., **8(2)**(2004), 235–244.

[12] Z. X. Chen and K. H. Shon, *On the entire function sharing one value CM with k-th derivatives*, J. Korean Math. Soc., **42(1)**(2005), 85–99.

[13] J. Clunie and W. K. Hayman, *The spherical derivative of integral and meromorphic functions*, Comment. Math. Helv., **40**(1966), 117–148.

[14] M. L. Fang and L. Zalcman, *Normal families and uniqueness theorems for entire functions*, J. Math. Anal. Appl., **280**(2003), 273–283.

[15] G. G. Gundersen, *Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates*, J. London Math. Soc., **37**(1998), 88–104.

[16] G. G. Gunedersen, *Finite order solution of second order linear differential equations*, Trans. Amer. Math. Soc., **305**(1998), 415–429.

[17] G. G. Gundersen and L.-Z. Yang, *Entire functions that share one value with one or two of their derivatives*, J. Math. Anal. Appl., **223(1)**(1998), 88–95.

[18] W. K. Hayman, *Meromorphic functions*, Clarendon Press,Oxford, 1964. Bull. London Math. Soc., **36**(2004),105–114.

[19] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo and J. L. Zhang, *Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity*, J. Math. Anal. Appl., **355**(2009), 352–363.

[20] J. Heittokangas, R. Korhonen and J. Rättyä, *Generalized logarithmic derivative estimates of Gol'dberg-Grinshtein type*,

[21] Z. B. Huang and R. R. Zhang, *Unqueness of the differences of meromorphic functions*, Anal. Math., **44(4)**(2018), 461–473.

[22] F. Marty, *Researches sur la ré partition des valeurs d'une mérmorphe*, Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys., **23**(1931), 183–261.

[23] E. Mues and N. Steimetz, *Meromorphe Funktionen die unit ihrer Ableitung Werte teilen*, Manuscripta Math., **29**(1979), 195–206.

[24] V. Ngoan and I. V. Ostrovski, *The logarithmic derivative of a meromorphic function*, Akad. Nauk Arjman. SSR Dokl., **41**(1965), 272–277.

[25] L. A. Rubel and C. C. Yang, *Values shared by an entire function and its derivative*, Complex analysis (Proc. Conf., Univ. Kentucky, Lexington, Ky.,1976), pp. 101–103, Lecture Notes in Math. **599**, Springer, Berlin, 1977.

[26] L. Yang, *Value Distribution Theory*, Springer-Verlag, Berlin, 1993.