UNIQUENESS OF TWO DIFFERENTIAL POLYNOMIALS OF A MEROMORPHIC FUNCTION SHARING A SET

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Abstract. In this paper, we are mainly devoted to find out the general meromorphic solution of some specific type of differential equation. We have also answered an open question posed by Banerjee-Chakraborty [4] by extending their results in a large extent. We have provided an example showing that the conclusion of the results of Zhang-Yang [16] is not general true. Some examples have been exhibited to show that certain claims are true in our main result. Finally some questions have been posed for the future research in this direction.

1. Introduction

In this paper, by a meromorphic function f, we mean a meromorphic function in the whole complex plane. We use the standard notation of Nevanlinna theory [5]. Let f and g be two non constant meromorphic functions in the complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f - a$ and $g - a$ have same set of zeros with the same multiplicities, we say that f and g share the value a CM (counting multiplicities) and if we do not consider the multiplicities, then f and g are said to share the value a IM (ignoring multiplicities).

When we use $a = \infty$, the zeros of $f - a$ means the poles of f.

In 1977, Rubel-Yang [11] first showed that Nevanlinna's Five Point Uniqueness Theorem can radically be improved if one considers the sharing of an entire function with its derivative. here we recall the result.

Theorem A ([11]). *If f is a non-constant entire function in the finite complex plane and if f and f' share two distinct finite values CM, then $f \equiv f'$.*

The following examples show that in Theorem A the number 'two' is the best possible.

Example 1.1 ([11]). Let

$$f(z) = e^{e^z} \int_0^z e^{-e^t}(1 - e^t)dt.$$

It is clear that $f' - 1 = e^z(f - 1)$. So f and f' share 1 CM but $f \not\equiv f'$.

Example 1.2. Let $f(z) = e^{-z} + 4$. It is clear that f and f' share the value 2 CM but $f \not\equiv f'$.

The next example shows that the same thing happens if 'entire' is replaced by 'non-entire meromorphic' function.

Example 1.3. Let $f(z) = \frac{2\mathcal{A}e^{2z}}{e^{2z} - \mathcal{B}}$. It is clear that f and f' share $\mathcal{A}$ IM but $f \not\equiv f'$.

In 1979, Mues-Steinmetz [10] further improved Theorem A as follows.

Theorem B ([10])**.** *Let f be a non-constant entire function. If f and f' share two distinct values a, b IM, then $f \equiv f'$.*

The following example shows that in Theorem B, one can not replace simply the word 'entire' by 'meromorphic'.

Example 1.4 ([10])**.** Let $f(z) = \left(\frac{1}{2} - \frac{\sqrt{5}i}{2} \tan\left(\frac{\sqrt{5}i}{4} z \right) \right)^2$. Clearly f and f' share 0, 1 IM, but note that $f \not\equiv f'$.

We next recall the following well known definition of set sharing.

Let S be a set of complex numbers and $E_f(S) = \bigcup_{a \in S}\{z : f(z) = a\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity, then the set $\bigcup_{a \in S}\{z : f(z) = a\}$ is denoted by $\overline{E}_f(S)$.

If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand, if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM. Evidently, if S contains only one element, then it coincides with the usual definition of CM (respectively, IM) sharing of values.

One can see from the following example that the result of Rubel-Yang or Mues-Steinmetzis are not in general true when we consider the sharing of a set of two elements instead of values.

Example 1.5. Let $\mathcal{S} = \left\{ \frac{2a}{5}, \frac{3a}{5} \right\}$, where $a(\neq 0)$ be any complex number. Let $f(z) = e^{-z} + a$, then $E_f(\mathcal{S}) = E_{f'}(\mathcal{S})$ but $f \not\equiv f'$.

So, for the uniqueness of an entire function and its derivative when sharing a set $\mathcal{S}$ CM, the cardinality of the range set should be at least three.

Example 1.6. Suppose $\mathcal{S} = \{-2i, 0, 2i\}$ and $f(z) = \frac{e^{2iz} - 1}{e^{iz}}$. Then f and f' share S IM but $f \not\equiv f'$.

Note 1.1. In Example 1.6, one may consider k-th derivative of f instead of first, when k is an odd positive integer.

Remark 1.1. So from Note 1.1, it is clear that for the uniqueness of an entire function f and its higher order derivative $f^{(k)}$ sharing a set S IM, the cardinality of a set S should be at least four.

To continue our discussions, we now define a small function as follows:

Definition 1.1. Let f be a non-constant meromorphic function. A function $a \equiv a(z)(\not\equiv 0, \infty)$ is said to be a small function of f if $T(r, a) = S(r, f)$.

In 2008, Yang-Zhang [12] obtained the following result.

Theorem C ([12])**.** *Let f be a non-constant meromorphic function and $q \geq 12$ be an integer. If f^q and $(f^q)'$ share 1 CM, then $f^q \equiv (f^q)'$, and assumes the form*

$$f(z) = ce^{\frac{z}{q}},$$

where c is a non-zero constant.

In 2009, Zhang-Yang [16] further improved Theorem C to a large extent by obtaining the following result.

Theorem D ([16])**.** *Let f be a non-constant meromorphic function, q, k be positive integers and $a \equiv a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $f^q - a$ and $(f^q)^{(k)} - a$ share the value 0 CM and $q > k + 1 + \sqrt{k+1}$, then $f^q \equiv (f^q)^{(k)}$ and f assumes the form*

$$f(z) = ce^{\frac{\lambda}{q}z},$$

where c is a non-zero constant and $\lambda^k = 1$.

Theorem E ([16])**.** *Let f be a non-constant meromorphic function, q, k be positive integers and $a \equiv a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $f^q - a$ and $(f^q)^{(k)} - a$ share the value 0 IM and $q > 2k + 3 + \sqrt{2k+3}$, then $f^q \equiv (f^q)^{(k)}$ and f assumes the form*

$$f(z) = ce^{\frac{\lambda}{q}z},$$

where c is a non-zero constant and $\lambda^k = 1$.

Remark 1.2. In the conclusion of Theorems D and E, for the case of higher order derivative $k \geq 2$, one may observe that, it is not always true the fact that if $f^q \equiv (f^q)^{(k)}$, then f assumes the form

$$f(z) = ce^{\frac{\lambda}{q}z}$$

with $c \in \mathbb{C}^*$ and $\lambda^k = 1$.

Following is a supportive example of the above observations.

Example 1.7. We choose f in such a way that $f^{13} = c_1 e^z + c_2 e^{\omega z} + c_3 e^{\omega^2 z}$, where ω is a non-real cube root of unity and $c_i \in \mathbb{C}^*$. Let $q = 13$, $k = 3$, then it is clear that $q > k + 1 + \sqrt{k+1}$ and $q > 2k + 3 + \sqrt{2k+3}$ and also $f^q \equiv (f^q)^{(k)}$ but

$$f(z) \neq ce^{\frac{\lambda}{q}z}$$

for a $c \in \mathbb{C}^*$ and $\lambda^k = 1$.

To continue the discussion we now recall the following definitions.

Definition 1.2 ([7]). Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.

    (i) $N(r, a; f \mid \geq p)$ $(\overline{N}(r, a; f \mid \geq p))$ denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.

    (ii) $N(r, a; f \mid \leq p)$ $(\overline{N}(r, a; f \mid \leq p))$ denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.

Definition 1.3 ([13]). For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid \geq 2) + \cdots + \overline{N}(r, a; f \mid \geq p)$. It is clear that $N_1(r, a; f) = \overline{N}(r, a; f)$.

From the above results, we see that the research about the uniqueness of a meromorphic function and its derivative sharing a 'value' or 'small function' has a long history. The best result obtained so far is for a function and its first derivative sharing a set with three elements. We also see that generally in the conclusion of the above results discussed so far, different possible forms of the function f have been exhibited. To serve the purpose researchers sometimes resorted to additional suppositions. But no attempt have so far been made to find the uniqueness of an entire or meromorphic function with its higher order derivatives sharing a set.

We now recall the following uniqueness polynomial introduced by Lin-Yi [8]

$$(1.1) \qquad P(w) = aw^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2,$$

where $n \geq 3$ is an integer and a, $b \in \mathbb{C}^*$ satisfying $ab^{n-2} \neq 2$. It is easy to verify that the polynomial $P(w)$ has only simple zeros.

We now recall the notion of weighted sharing which is a scaling between CM or IM sharing of values or sets appeared first in the literature in 2001 [6].

Definition 1.4 ([6]). Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_f(a, k)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_f(a, k) = E_g(a, k)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.5 ([6]). Let $\mathcal{S}$ be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(\mathcal{S}, k)$ the set $\bigcup_{a \in \mathcal{S}} E_k(a; f)$. If $E_f(\mathcal{S}, k) = E_g(S, k)$, then we say f, g share the set $\mathcal{S}$ with weight k.

Very recently, in this direction, with the help of weighted sharing of sets Banerjee-Chakraborty [4] considered a homogeneous differential polynomial

$$L(f) = a_0 \left(f^{(k)} \right)^l + a_1 \left(f^{(k-1)} \right)^l + \cdots + a_{k-1} \left(f' \right)^l,$$

where $l, k \in \mathbb{N}$, $a_i \in \mathbb{C}$ and obtained the following result.

Theorem F ([4])**.** *Let $m(\geq 1)$, $n(\geq 1)$ be positive integers and f be a non-constant meromorphic function. Suppose $\mathcal{S} = \{w : P(w) = 0\}$ and $E_{f^m}(\mathcal{S}, p) = E_{L(f)}(\mathcal{S}, p)$. If one of the following conditions holds*:

(1) $2 \leq p < \infty$ *and* $n > 6 + \frac{6(\mu+1)}{\lambda-2\mu}$, *or*

(2) $p = 1$ *and* $n > \frac{13}{2} + \frac{7(\mu+1)}{\lambda-2\mu}$, *or*

(3) $p = 0$ *and* $n > 6 + 3\mu + \frac{6(\mu+1)^2}{\lambda-2\mu}$,

then $f^m \equiv L(f)$, *where* $\lambda = \min\{m(n-2) - 1, l(k+1)(n-2) - 1\}$ *and* $\mu = \min\left\{\frac{1}{p}, 1\right\}$.

Note 1.2. One may observe that one can not find the lower bound n for the case $p = 0$ in Theorem F since the number μ is undefined in the case (3) when $p = 0$.

So a natural question arises as follows:

Question 1.1. Is it possible in anyway, to find a corresponding result of Theorem F for the case $p = 0$ by removing the difficulty mentioned above?

In [4], Banerjee-Chakrabarty asked an open question as follows:

Question 1.2. Can the conclusion of Theorem F remain valid if any non-homogeneous differential polynomial generated by f is considered?

Therefore to extend all the above theorems in a large extent and also to answer the above questions, we are now at a position to define differential polynomial as follows.

Definition 1.6 ([3])**.** Let $n_{0j}, n_{1j}, \ldots, n_{kj}$ be non-negative integers. Also let $g = f^q$.
• The expression $M_j[g] = (g)^{n_{0j}} (g')^{n_{1j}} \cdots (g^{(k)})^{n_{kj}}$ is called a differential monomial generated by g of degree $d(M_j) = \sum_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j} = \sum_{i=0}^{k} (1 + i)n_{ij}$.
• The sum $P[g] = \sum_{j=1}^{t} b_j M_j[g]$ is called a differential polynomial generated by g of degree $\bar{d}(P) = \max\{d(M_j) : 1 \leq j \leq t\}$ and weight $\Gamma_{\mathcal{P}} = \max\{\Gamma_{M_j} : 1 \leq j \leq t\}$, where $T(r, b_j) = S(r, g)$ for $j = 1, 2, \ldots, t$.
• The numbers $\underline{d}(P) = \min\{d(M_j) : 1 \leq j \leq t\}$ and k the highest order of the derivative of g in $P[g]$ are called respectively the lower degree and order of $P[g]$.
• $P[g]$ is called *homogeneous* if $\bar{d}(P) = \underline{d}(P)$.
• $P[g]$ is called a *linear differential polynomial* generated by g if $\bar{d}(P) = 1$. Otherwise $P[g]$ is called *non-linear differential polynomial*. We denote by $Q = \max\{\Gamma_{M_j} - d(M_j) : 1 \leq j \leq t\}$.

Next we have the following observation.

Note 1.3. For a non-constant meromorphic function f, let us suppose that $f \equiv f^{(k)}$. It is clear that f can not have any pole. Again since no non-constant polynomial satisfies the relation, so it is very natural that f must be a transcendental entire function. So one must have the general solution of $f \equiv f^{(k)}$ as follows:

(1) When $N(r, 0; f) \neq S(r, f)$, then

$$f(z) = c_0 \exp(z) + c_1 \exp(\theta z) + c_2 \exp(\theta^2 z) + \cdots + c_{k-1} \exp(\theta^{k-1} z),$$

(2) When $N(r, 0; f) = S(r, f)$, then

$$f(z) = c \exp(\theta z),$$

where $\theta = \cos\left(\frac{2\pi}{k}\right) + i \sin\left(\frac{2\pi}{k}\right)$ and $c(\neq 0)$, $c_{i-1} \in \mathbb{C}$, not all zero, for $i \in \{1, 2, \ldots, k\}$.

So observing Note 1.3, one natural question arises as follows:

Question 1.3. Can we extend $f^{(k)}$ up to a general differential monomial $M[f]$ to get a certain form of the function which satisfies the relation $f \equiv M[f]$?

The answer of Question 1.3 is not true in general. Suppose $M[f] = f^{(k)} f^{(r)}$ or $\left(f^{(k)}\right)^{n_k} \left(f^{(r)}\right)^{n_r}$ etc., where k, r and n_k, n_r all are positive integers with $k > r$. We see that the form of the function in Note 1.3 does not satisfy the relation $f \equiv M[f]$.

Since our main motivation is to extend $f^{(k)}$ up to a general differential monomial $M[f]$, and also to find a non-constant meromorphic solution of the relation $f \equiv M[f]$, so the worth noticing fact here is that, we need some power in the first setting of the relation. If so, then the question arises: '*does it really help us to get $f^p \equiv M[f]$ for the function in* Note 1.3*?*' The answer is NO in general. We explain the fact in the following.

(i) Suppose that $f(z) = c_1 \exp(z) + c_2 \exp(-z)$ and $M[f] = \left(f^{(k)}\right)^{n_k} \left(f^{(r)}\right)^{n_r}$, where $p = n_k + n_r$, k and r are even positive integers. In this case, one can easily obtained $f^p \equiv M[f]$.

(ii) But, if $f(z) = c_1 \exp(z) + c_2 \exp(-z)$ and $M[f] = \left(f^{(k)}\right)^{n_k} \left(f^{(r)}\right)^{n_r}$, where one of k and r is even and other is odd positive integer. Then $f^p \not\equiv M[f]$ for all positive integer p.

Thus, we have the following observation.

Note 1.4. For a more general setting $f^{d_M} \equiv M[f]$, where

$$M[f] = (f)^{n_0} (f')^{n_1} (f'')^{n_2} \cdots \left(f^{(k)}\right)^{n_k},$$

we see that $f(z) = c \exp(\lambda z)$, $\lambda^{Q_M} = 1$, is a certain solution of it, where $Q_M = \Gamma_M - d_M$, $\Gamma_M = \sum_{i=0}^{k}(i+1)n_i$ and $d_M = \sum_{i=0}^{k} n_i$.

The above observations motivate oneself to construct a new setting in the place of f or f^{d_M} which when shares a set $\mathcal{S}$ with its differential polynomial

$P[f]$ to get $f(z) = c\exp(\lambda z)$ as a solution of the identical relation between them.

So the following question is inevitable.

Question 1.4. What setting one should assume in the place of f or in f^{d_M} to get $f(z) = c\exp(\lambda z)$ as a certain solution of the identical relation of that setting with $P[f]$?

To this end, next we define

$$\Im(z) = \sum_{j=1}^{t} b_j z^{d_{M_j}} = b_1 z^{d_{M_1}} + b_2 z^{d_{M_2}} + \cdots + b_t z^{d_{M_t}},$$

where $b_j (j = 1, 2, \ldots, t)$ are all constants.

Since the natural extension of the derivatives of a meromorphic function is differential monomials and hence differential polynomials generated by f, so for the improvements as well as extensions in this direction further, the following questions are inevitable.

Question 1.5. Is it possible that power of a meromorphic function when sharing a set together with its k-th derivative or differential monomial or even its differential polynomial becomes identical?

If the answer of the above question is affirmative, then another natural question arises as follows.

Question 1.6. Is it possible in anyway to get a solution or sometimes a specific form of the function of the identical relation?

Answering all the above mentioned questions affirmatively is the main motivation of writing this paper.

2. Main results

Following theorems are the main results of this paper which answer all the above questions and the query mentioned in Note 1.4 affirmatively.

Theorem 2.1. *Let n, k, q $(\geq k+1) \in \mathbb{N}$ and f be a non-constant meromorphic function. Suppose that $\mathcal{S} = \{w : P(w) = 0\}$. If $E_{\Im(f^q)}(\mathcal{S}, p) = E_{P[f^q]}(\mathcal{S}, p)$ and one of the following conditions holds*:

(1) $2 \leq p < \infty$ *and* $n > \max\left\{2\overline{d}(P), \; 5 + \frac{6(2p^2 + p\gamma_m - 2)}{(p\delta - 2)(p\gamma_m - 2) - p(2+p)}\right\}$,

(2) $p = 1$ *and* $n > \max\left\{2\overline{d}(P), \; \frac{11}{2} + \frac{\gamma_m}{(\delta - 2)(\gamma_m - 2) - 3}\right\}$,

(3) $p = 0$ *and* $n > \max\left\{2\overline{d}(P), \; 11 + \frac{12\gamma_m}{(\delta - 2)(\gamma_m - 2) - 3}\right\}$,

then

$$\Im(f^q) \equiv P[f^q],$$

where $\delta = (n-2)\bar{d}(P) - 1$ and $\gamma_m = \min_{1 \le j \le t}\left\{2d_{M_j} - \Gamma_{M_j}\right\}n - 1$.

Furthermore, we see that the function f of the form

$$f(z) = c\,\exp\left(\frac{\lambda}{q}z\right),$$

where c is a non-zero constant with $\lambda^{\Gamma_{Q_j}} = 1$ for all $j = 1, 2, \ldots, t$, is a certain solution of $\Im(f^q) \equiv P[f^q]$.

Theorem 2.2. *Let n, k, q $(\ge k+1) \in \mathbb{N}$ and f be a non-constant entire function. Suppose that $\mathcal{S} = \{w : P(w) = 0\}$ and $E_{\Im(f^q)}(\mathcal{S}, p) = E_{P[f^q]}(\mathcal{S}, p)$. If one of the following conditions holds:*

(1) $2 \le p < \infty$ *and* $n > \max\left\{2\bar{d}(P),\ 5\right\}$,
(2) $p = 1$ *and* $n > \max\left\{2\bar{d}(P),\ \frac{11}{2}\right\}$,
(3) $p = 0$ *and* $n > \max\left\{2\bar{d}(P),\ 11\right\}$,

then conclusions of Theorem 2.1 hold.

Remark 2.1. Theorem 1.1 directly improves Theorem F by extending from homogeneous differential polynomial to non-homogeneous differential polynomial as well as by reducing the lower bound of the cardinality of the set.

Next in particular, if we consider $\Im(f^q) = f^q$ and $P[f^q] = (f^q)^{(k)}$, then clearly $\bar{d}(P) = 1$ and $\Gamma_P = k$. So, we have the following corollaries.

Corollary 2.1. *Let n, k, q $(\ge k+1) \in \mathbb{N}$ and f be a non-constant entire function such that $N(r, 0; f) = S(r, f)$. Suppose that $\mathcal{S} = \{w : P(w) = 0\}$, and $E_{f^q}(\mathcal{S}, p) = E_{(f^q)^{(k)}}(\mathcal{S}, p)$. If one of the following conditions holds:*

(1) $2 \le p < \infty$ *and* $n > 2$,
(2) $p = 1$ *and* $n > 2$,
(3) $p = 0$ *and* $n > 4$,

then $f^q \equiv (f^q)^{(k)}$.

Then f assume the form

$$f(z) = c\,\exp\left(\frac{\lambda}{q}z\right),$$

where c is a non-zero constant and $\lambda^k = 1$.

Remark 2.2. From Corollary 2.1, we see that for the uniqueness of f and its k-th derivative sharing $(\mathcal{S}, 1)$ when $N(r, 0; f) = S(r, f)$, the cardinality of the set $\mathcal{S}$ is 3.

Remark 2.3. We see from Corollary 2.1, for the uniqueness of f and its k-th derivative sharing $(\mathcal{S}, 0)$ when $N(r, 0; f) = S(r, f)$, the cardinality of the set $\mathcal{S}$ is 5.

Remark 2.4. From Theorems 2.1 and 2.2, we observe that there exists a set $\mathcal{S}$ with 6 elements such that $E_{f^q}(\mathcal{S}, 3) = E_{(f^q)^{(k)}}(\mathcal{S}, 3)$ with q $(\ge k+1)$, then $f^q \equiv (f^q)^{(k)}$ for a non constant meromorphic function f.

The following two examples show that for a non-constant entire function the set $\mathcal{S}$ in Theorem 2.1 can not be replaced by an arbitrary set containing six distinct elements.

Example 2.1. Let $\mathcal{S} = \{-6, -6\omega^2, 0, 2\omega, 4\omega, 6\omega\}$, where ω is the non-real cubic root of unity. Let k be an odd positive integer. Choosing $f(z) = (e^{-z} + 6\omega)^{\frac{1}{q}}$, where $q \geq k + 1$. It is easy to verify that $\Im(f^q) = f^q$ and $P[f^q] = (f^q)^{(k)} - (f^q)'''(f^q)' + ((f^q)'')^2$ share $(\mathcal{S}, \infty)$, but $\Im(f^q) \not\equiv P[f^q]$.

Example 2.2. Let k be a positive integer and θ be a root of the equation $z^k + 1 = 0$. Let $f(z) = (e^{\theta z} + a)^{\frac{1}{q}}$, where $q \geq k + 1$ and a is a non-zero complex number. Let $S = \left\{\frac{a}{6}, \frac{5a}{6}, \frac{3a}{7}, \frac{4a}{7}, \frac{2a}{9}, \frac{7a}{9}\right\}$. Therefore it is easy to verify that $\Im(f^q) = f^q$ and $P[f^q] = (f^q)^{(k)}$ share $(\mathcal{S}, \infty)$, but $\Im(f^q) \not\equiv P[f^q]$.

The next example shows that for a non-constant entire function the set S in Corollary 2.1 can not be replaced by an arbitrary set containing three distinct elements.

Example 2.3. For a non-zero constant a, let $\mathcal{S} = \{a, a\omega, a\omega^2\}$, where ω is the non-real cube root of unity. Choosing $f(z) = \exp\left(\frac{1}{q}\omega^{\frac{2}{k}} z\right)$, it is easy to verify that $N(r, 0; f) = S(r, f)$. Note that f^q and $(f^q)^{(k)}$ share $(\mathcal{S}, \infty)$, but $f^q \not\equiv (f^q)^{(k)}$.

3. Some lemmas

We define $\mathcal{R}(w) = \frac{aw^n}{n(n-1)(w-\alpha_1)(w-\alpha_2)}$, where $\alpha_i, (i = 1, 2)$ are the distinct roots of the equation

$$n(n-1)w^2 - 2n(n-2)bw + (n-1)(n-2)b^2 = 0.$$

Then

$$\mathcal{R}(w) - 1 = \frac{P(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)}.$$

Let $\mathcal{F} = \mathcal{R}(\Im(f^q))$ and $\mathcal{G} = \mathcal{R}(P[f^q])$, where $q \geq k + 1$ and $f(z)$ is a non-constant meromorphic function and associated with $\mathcal{F}$ and $\mathcal{G}$, we define

$$\mathcal{H} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F}-1}\right) - \left(\frac{\mathcal{G}''}{\mathcal{G}'} - \frac{2\mathcal{G}'}{\mathcal{G}-1}\right).$$

Lemma 3.1 ([14]). *Let f be a non-constant meromorphic function and $\mathcal{Q}(f) = a_m f^q + a_{m-1} f^{m-1} + \cdots + a_0$, where $a_0, a_1, \ldots, a_m$ are constants with $a_m \neq 0$. Then*

$$T(r, \mathcal{Q}(f)) = m\,T(r, f) + S(r, f).$$

Lemma 3.2 ([15]). *Let h be a non-constant meromorphic function, and let a_j be distinct finite complex numbers such that $a_j \neq 0$ for $j = 1, 2, \ldots, q$. Then*

$$\sum_{j=1}^{q}\left(N\left(r, \frac{1}{h-a_j}\right) - \overline{N}\left(r, \frac{1}{h-a_j}\right)\right) \leq \overline{N}(r, 0; h) + \overline{N}(r, \infty; h) + S(r, h).$$

Lemma 3.3 ([13]). *Let f be a non-constant meromorphic function. Then*

$$N\left(r, 0; f^{(k)}\right) \leq N(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f).$$

Lemma 3.4. *Let $\mathcal{F}$ and $\mathcal{G}$ share $(1, p)$ where $\mathcal{F}$ and $\mathcal{G}$ defined as earlier. Then*

(1) $\overline{N}_L(r, 1; \mathcal{F}) \leq \left\{\overline{N}(r, 0; \Im(f^q)) + \overline{N}(r, \infty; f)\right\} + S(r, f)$ *for $p = 0$.*

(2) $\overline{N}_L(r, 1; \mathcal{F}) \leq \frac{1}{p}\left\{\overline{N}(r, 0; \Im(f^q)) + \overline{N}(r, \infty; f)\right\} + S(r, f)$ *for $p \geq 1$.*

(3) $\overline{N}_L(r, 1; \mathcal{G}) \leq \left\{\overline{N}(r, 0; P[f^q]) + \overline{N}(r, \infty; f)\right\} + S(r, P[f^q])$ *for $p = 0$.*

(4) $\overline{N}_L(r, 1; \mathcal{G}) \leq \frac{1}{p}\left\{\overline{N}(r, 0; P[f^q]) + \overline{N}(r, \infty; f)\right\} + S(r, P[f^q])$ *for $p \geq 1$.*

Proof. First we note that in view of Lemma 3.1, we get $S(r, \Im(f)) = S(r, f)$. By using Lemma 3.2, we obtained when $p = 0$,

$$\begin{aligned}
\overline{N}_L(r, 1; \mathcal{F}) &\leq N(r, 1; \mathcal{F}) - \overline{N}(r, 1, \mathcal{F}) \\
&\leq \overline{N}(r, 0; \Im(f^q)) + \overline{N}(r, \infty; \Im(f^q)) + S(r, f) \\
&\leq \overline{N}(r, 0; \Im(f^q)) + \overline{N}(r, \infty; f) + S(r, f).
\end{aligned}$$

When $p \geq 1$, we get by using Lemma 3.2

$$\begin{aligned}
\overline{N}_L(r, 1; \mathcal{F}) &\leq \overline{N}(r, 1; \mathcal{F}| \geq p + 1) \\
&\leq \frac{1}{p}\left\{N(r, 1; \mathcal{F}) - \overline{N}(r, 1, \mathcal{F})\right\} \\
&\leq \frac{1}{p}\left\{\overline{N}(r, 0; \Im(f^q)) + \overline{N}(r, \infty; \Im(f^q))\right\} + S(r, f) \\
&\leq \frac{1}{p}\left\{\overline{N}(r, 0; \Im(f^q)) + \overline{N}(r, \infty; f)\right\} + S(r, f).
\end{aligned}$$

Combining the two cases we get the proof.

Similarly we can prove the other one.   □

Lemma 3.5. *Suppose that $\mathcal{F}$ and $\mathcal{G}$ share $(1, p)$, where $0 \leq p < \infty$. If $\mathcal{F} \not\equiv \mathcal{G}$ and $q \geq k + 1$, then*

(1) *for $p \geq 1$,*

$$\begin{aligned}
\overline{N}(r, 0; f) &\leq \overline{N}(r, 0; P[f^q]) \\
&\leq \frac{2 + p}{p\gamma_m - 2}\overline{N}(r, \infty; f) + \frac{2p}{p\gamma_m - 2}\left(T(r, \Im(f^q)) + T(r, P[f^q])\right).
\end{aligned}$$

(2) *for $p = 0$,*

$$\begin{aligned}
\overline{N}(r, 0; f) &\leq \overline{N}(r, 0; P[f^q]) \\
&\leq \frac{3}{\gamma_m - 2}\overline{N}(r, \infty; f) + \frac{2}{\gamma_m - 2}\left(T(r, \Im(f^q)) + T(r, P[f^q])\right),
\end{aligned}$$

where $\gamma_m = \min_{1 \leq j \leq t}\left\{2d_{M_j} - \Gamma_{M_j}\right\}n - 1$.

Proof. We define $\Phi = \frac{\mathcal{F}'}{\mathcal{F}-1} - \frac{\mathcal{G}'}{\mathcal{G}-1}$. We now split the problem in two cases as follows:

Case 1. Suppose that $\Phi \equiv 0$. Then by integration, we have

$$\mathcal{F} - 1 \equiv \mathcal{B}(\mathcal{G} - 1).$$

Let if possible z_0 be a zero of f, then $\mathcal{B} = 1$ which contradicts $\mathcal{F} \not\equiv \mathcal{G}$. Thus we get $\overline{N}(r, 0; f) = S(r, f)$ and the result is hold.

Case 2. So, we suppose that $\Phi \not\equiv 0$. Let us suppose that z_0 be a zero of f of order s, then z_0 would be a zero of $\mathcal{F}$ of order $sq\underline{d}(P)n$ and from [2, Lemma 2.5], we see that z_0 be a zero of $\mathcal{G}$ of order $\min_{1 \leq j \leq t} \left\{(s+1)d_{M_j} - \Gamma_{M_j}\right\} n$. Therefore, it is clear that z_0 is a zero of Φ of order at least

$$\min \left\{ q\underline{d}(P)n - 1, \min_{1 \leq j \leq t} \left\{ 2d_{M_j} - \Gamma_{M_j} \right\}n - 1 \right\}$$

$$= \min_{1 \leq j \leq t} \left\{ 2d_{M_j} - \Gamma_{M_j} \right\} n - 1$$

$$= \gamma_m \ (say).$$

When $p \geq 1$, then using Lemma 3.4, we have

$$\overline{N}(r, 0; f)$$

$$\leq \overline{N}(r, 0; P[f^q])$$

$$\leq \frac{1}{\gamma_m} N(r, 0; \Phi)$$

$$\leq \frac{1}{\gamma_m} N(r, \infty; \Phi) + S(r, P[f^q])$$

$$\leq \frac{1}{\gamma_m} \left\{ \overline{N}_L(e, 1; \mathcal{F}) + \overline{N}_L(e, 1; \mathcal{G}) + \overline{N}_L(e, \infty; \mathcal{F}) + \overline{N}_L(e, \infty; \mathcal{G}) \right.$$

$$\left. + \overline{N}_L(e, \infty; \mathcal{F}|\mathcal{G} \neq \infty) + \overline{N}_L(e, \infty; \mathcal{G}|\mathcal{F} \neq \infty) \right\} + S(r, P[f^q])$$

$$\leq \frac{1}{\gamma_m} \left\{ \frac{1}{p} \left[\overline{N}(r, 0; \Im(f^q)) + \overline{N}(r, 0; P[f^q]) + 2\overline{N}(r, \infty; f) \right] + \overline{N}(r, \infty; f) \right.$$

$$\left. + \sum_{i=1}^{2} \left(\overline{N}(r, \alpha_i; \Im(f^q)) + \overline{N}(r, \alpha_i; P[f]) \right) \right\} + S(r, P[f^q])$$

$$\leq \frac{1}{\gamma_m} \left\{ \frac{2}{p} \overline{N}(r, 0; P[f^q]) + \left(\frac{2}{p} + 1 \right) \overline{N}(r, \infty; f) + 2 \left\{ T(r, \Im(f^q)) \right. \right.$$

$$\left. \left. + T(r, P[f^q]) \right\} \right\} + S(r, P[f^q]),$$

i.e.,

$$\overline{N}(r, 0; f) \leq \overline{N}(r, 0; P[f^q])$$

$$\leq \frac{2+p}{p\gamma_m - 2}\overline{N}(r,\infty;f) + \frac{2p}{p\gamma_m - 2}\Big\{T(r,\Im(f^q)) + T(r,P[f^q])\Big\}.$$

When $p = 0$, then proceeding exactly same way as done in above, we get

$$\overline{N}(r,0;f) \leq \overline{N}(r,0;P[f^q])$$

$$\leq \frac{3}{\gamma_m - 2}\overline{N}(r,\infty;f) + \frac{2}{\gamma_m - 2}\Big\{T(r,\Im(f^q)) + T(r,P[f^q])\Big\}. \qquad \square$$

Lemma 3.6. *Let* $\mathcal{F}$ *and* $\mathcal{G}$ *share* $(1,p)$, *where* $\mathcal{F}$ *and* $\mathcal{G}$ *defined as earlier. If* $\mathcal{F} \not\equiv \mathcal{G}$, *then*
  (1) *for* $p = 0$,

(3.1) $\overline{N}(r,\infty;f)$

$$\leq \frac{\gamma_m}{(\delta-2)(\gamma_m-2)-3}\Big\{T(r,\Im(f^q)) + T(r,P[f^q])\Big\}$$
$$+ S(r,P[f^q]) + S(r,f),$$

  (2) *for* $p \geq 1$,

(3.2) $\overline{N}(r,\infty;f)$

$$\leq \frac{2p^2+p\gamma_m-2}{(p\delta-2)(p\gamma_m-2)-p(2+p)}\Big\{T(r,\Im(f^q)) + T(r,P[f^q])\Big\}$$
$$+ S(r,P[f^q]) + S(r,f),$$

where $\delta = (n-2)\overline{d}(P) - 1$ *and* $\gamma_m = \min_{1\leq j\leq t}\big\{2d_{M_j} - \Gamma_{M_j}\big\}n - 1$.

Proof. Let us define $\Psi = \frac{\mathcal{F}'}{\mathcal{F}(\mathcal{F}-1)} - \frac{\mathcal{G}'}{\mathcal{G}(\mathcal{G}-1)}$.
Case 1. Suppose $\Psi \equiv 0$.
  By integration, we get $\big(1 - \frac{1}{\mathcal{F}}\big) = \mathcal{A}\big(1 - \frac{1}{\mathcal{G}}\big)$. As $\Im(f^q)$ and $P[f^q]$ share $(\infty,0)$, so if $\overline{N}(r,\infty;f) \neq S(r,f)$ then $\mathcal{A} = 1$, i.e., $\mathcal{F} = \mathcal{G}$, which is not possible. So, $\overline{N}(r,\infty;f) = S(r,f)$. Thus the lemma holds.
Case 2. Let $\Psi \not\equiv 0$.
  Let z_0 be a pole of f of order r, then it is a pole of $\Im(f^q)$ of order $r\overline{d}(P)$ and of $P[f^q]$ of order $r\overline{d}(P) + \Gamma_P$ and that of $\mathcal{F}$ and $\mathcal{G}$ are $r\overline{d}(P)(n-2)$ and $(r\overline{d}(P) + \Gamma_P)(n-2)$ respectively.
  Clearly z_0 is a zero of $\frac{\mathcal{F}'}{\mathcal{F}-1} - \frac{\mathcal{F}'}{\mathcal{F}}$ and $\frac{\mathcal{G}'}{\mathcal{G}-1} - \frac{\mathcal{G}'}{\mathcal{G}}$ of order at least $(n-2)\overline{d}(P)-1$ and $(\overline{d}(P)+\Gamma_P)(n-2)-1$ respectively and hence a zero of Ψ of order at least,

$$\min\{(n-2)\overline{d}(P) - 1, (\overline{d}(P) + \Gamma_P)(n-2) - 1\}$$
$$= (n-2)\overline{d}(P) - 1 = \delta \text{ (say)}.$$

Thus using Lemma 3.4, we get for $p \geq 1$,

$$\overline{N}(r,\infty;f)$$
$$\leq \frac{1}{\delta}N(r,0;\Psi)$$

$$\leq \frac{1}{\delta} N(r, \infty; \Psi) + S(r, P[f]) + S(r, f)$$

$$\leq \frac{1}{\delta}\left\{\overline{N}_L(r, 1; \mathcal{F}) + \overline{N}_L(r, 1; \mathcal{G}) + \overline{N}(r, 0; P[f^q])\right\} + S(r, P[f^q]) + S(r, f)$$

$$\leq \frac{1}{\delta}\left[\frac{1}{p}\left\{\overline{N}(r, 0; \Im(f^q)) + \overline{N}(r, \infty; f) + \overline{N}(r, 0; P[f^q]) + \overline{N}(r, \infty; f)\right\}\right.$$
$$\left. + \overline{N}(r, 0; P[f^q])\right] + S(r, P[f^q]) + S(r, f).$$

Thus using Lemma 3.5, we get

$$\left(\delta - \frac{2}{p}\right)\overline{N}(r, \infty; f)$$

$$\leq \overline{N}(r, 0; P[f^q]) + \frac{1}{p}\left\{T(r, \Im(f^q)) + T(r, P[f^q])\right\} + S(r, P[f^q]) + S(r, f),$$

i.e.,

$$\overline{N}(r, \infty; f)$$

$$\leq \frac{p}{p\delta - 2}\overline{N}(r, 0; P[f^q]) + \frac{1}{p\delta - 2}\left\{T(r, \Im(f^q)) + T(r, P[f^q])\right\} + S(r, P[f^q])$$
$$+ S(r, f)$$

$$\leq \frac{p(2 + p)}{(p\delta - 2)(p\gamma_m - 2)}\overline{N}(r, \infty; f) + \left[\frac{2p^2}{(p\delta - 2)(p\gamma_m - 2)} + \frac{1}{p\delta - 2}\right]$$
$$\times \left\{T(r, \Im(f^q)) + T(r, P[f^q])\right\} + S(r, P[f^q]) + S(r, f),$$

i.e.,

$$\overline{N}(r, \infty; f)$$

$$\leq \frac{2p^2 + p\gamma_m - 2}{(p\delta - 2)(p\gamma_m - 2) - p(2 + p)}\left\{T(r, \Im(f^q)) + T(r, P[f^q])\right\} + S(r, P[f^q])$$
$$+ S(r, f).$$

Next for $p = 0$, using Lemmas 3.4, 3.5 and proceeding exactly as above we get

$$\overline{N}(r, \infty; f)$$

$$\leq \frac{\gamma_m}{(\delta - 2)(\gamma_m - 2) - 3}\left\{T(r, \Im(f^q)) + T(r, P[f^q])\right\} + S(r, P[f^q]) + S(r, f).$$

This completes the proof. $\qquad\square$

Lemma 3.7. *If $\mathcal{H} \not\equiv 0$ and $\mathcal{F}$ and $\mathcal{G}$ share $(1, p)$, then*

$$(3.3) \qquad N(r, \infty; \mathcal{H})$$

$$\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; \Im(f^q)) + \overline{N}(r, 0; P[f^q]) + \overline{N}(r, b; \Im(f^q))$$
$$+ \overline{N}(r, b; P[f^q]) + \overline{N}_L(r, 1; \mathcal{F}) + \overline{N}_L(r, 1; \mathcal{G}) + \overline{N}_0(r, 0; (\Im(f^q))')$$

$$+ \overline{N}_0(r, 0; (P[f^q])'),$$

where $\overline{N}_0(r, 0; (\Im(f^q))')$ denotes the counting function of all those zeros of $(\Im(f^q))'$ which are not the zeros of $\Im(f^q)(\Im(f^q) - b)$ and $\mathcal{F} - 1$. Similar expressions holds for $P[f^q]$.

Proof. If we use the following fact, then the proof will be easy. A zero of f^q may not be a zero of $P[f^q]$ but an elementary calculations shows that when $q \geq k+1$, then each zeros of f^q must be a zero of $P[f^q]$, so we have $\overline{N}(r, 0; \mathcal{F}) \leq \overline{N}(r, 0; \mathcal{G})$. Also we note that $\overline{N}(r, \infty; \mathcal{F}) \leq \overline{N}(r, \infty; f) + \overline{N}(r, \alpha_1; \Im(f^q)) + \overline{N}(r, \alpha_2; \Im(f^q))$. But note that the simple zeros of $\Im(f^q) - \alpha_i$ are not the poles of $\mathcal{H}$ and multiple zeros of $\Im(f^q) - \alpha_i$ are zeros of $(\Im(f^q))'$. Similar explanations hold for $\mathcal{G}$ also. $\square$

Lemma 3.8 ([1]). *Let* $\phi(w) = (n-1)^2(w^{n-2}-1)(w^n-1) - n(n-2)(w^{n-1}-1)^2$. *Then*

$$\phi(w) = (w-1)^4 \prod_{i=1}^{2n-6} (w - \beta_i),$$

where $\beta_i \in \mathbb{C}^* - \{1\}, (i = 1, 2, \ldots, 2n-6)$, which are distinct.

4. Proofs of the theorems

Proof of Theorem 2.1. We split the whole proof into two different cases as follows.
Case 1. In this case, we assume that $\mathcal{H} \not\equiv 0$. So, one can see that $\mathcal{F} \not\equiv \mathcal{G}$.
   We note that $\overline{N}(r, 1; \mathcal{F}| = 1) = \overline{N}(r, 1; \mathcal{G}| = 1) \leq N(r, \infty; \mathcal{H})$.
   By using the Second Fundamental Theorem and Lemma 3.4, we get

(4.1)
$$(n+1)T(r, \Im(f^q))$$
$$\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; \Im(f^q)) + \overline{N}(r, b; \Im(f^q)) + \overline{N}(r, 1; \mathcal{F})$$
$$\quad - N_0(r, 0, (\Im(f^q))') + S(r, f)$$
$$\leq 2\left\{ \overline{N}(r, \infty; f) + \overline{N}(r, b; \Im(f^q)) \right\} + \left\{ \overline{N}(r, 0; \Im(f^q)) + \overline{N}(r, 0; P[f^q]) \right.$$
$$\quad + \overline{N}(r, b; P[f^q]) \right\} + \left\{ \overline{N}(r, 1; \mathcal{F}| \geq 2) + \overline{N}_L(r, 1; \mathcal{F}) + \overline{N}_L(r, 1; \mathcal{G}) \right.$$
$$\quad + \overline{N}_0(r, 0; (P[f^q])') \right\} + S(r, f).$$

Subcase 1.1. When $p \geq 2$. Then we have the following

(4.2)
$$\overline{N}(r, 1; \mathcal{F}| \geq 2) + \overline{N}_L(r, 1; \mathcal{F}) + \overline{N}_L(r, 1; \mathcal{G}) + \overline{N}_0(r, 0; (P[f])')$$
$$\leq \overline{N}(r, 1; \mathcal{G}| \geq 2) + \overline{N}(r, 1; \mathcal{G}| \geq 3) + \overline{N}_0(r, 0; (P[f])')$$
$$\leq N\left(r, 0; (P[f])' \middle| P[f] \neq 0\right) + S(r, P[f])$$

$$\leq N\left(r, \infty; \frac{(P[f^q])'}{P[f^q]}\right) + S(r, P[f^q])$$

$$\leq \overline{N}(r, 0; P[f^q]) + \overline{N}(r, \infty; f) + S(r, P[f^q]).$$

With the help of this, note that (4.1) becomes

(4.3) $(n+1)T(r, \Im(f^q))$

$$\leq 2\left\{\overline{N}(r, \infty; f) + \overline{N}(r, b; \Im(f^q))\right\} + 2\overline{N}(r, 0; P[f^q])$$

$$+ \overline{N}(r, 0; \Im(f^q)) + \overline{N}(r, b; P[f^q]) + \overline{N}(r, \infty; f) + S(r, P[f^q]) + S(r, f).$$

Similarly for $P[f^q]$, we get

(4.4) $(n+1)T(r, P[f^q])$

$$\leq 2\left\{\overline{N}(r, \infty; f) + \overline{N}(r, b; P[f^q])\right\} + 2\overline{N}(r, 0; \Im(f^q))$$

$$+ \overline{N}(r, 0; P[f^q]) + \overline{N}(r, b; \Im(f^q)) + \overline{N}(r, \infty; f) + S(r, P[f^q]) + S(r, f).$$

Adding (4.3) and (4.4), we obtained

(4.5) $$(n+1)\left\{T(r, \Im(f^q)) + T(r, P[f^q])\right\}$$

$$\leq 6\overline{N}(r, \infty; f) + 3\left\{\overline{N}(r, 0; \Im(f^q)) + \overline{N}(r, 0; P[f^q])\right\}$$

$$+ 3\left\{\overline{N}(r, b; \Im(f^q)) + \overline{N}(r, b; P[f^q])\right\} + S(r, P[f^q]) + S(r, f),$$

i.e.,

(4.6) $$(n-5)\left\{T(r, \Im(f^q)) + T(r, P[f^q])\right\}$$

$$\leq 6\overline{N}(r, \infty; f) + S(r, P[f^q]) + S(r, f).$$

Using Lemma 3.6 in (4.6), we get

$$(n-5)\left\{T(r, \Im(f)) + T(r, P[f^q])\right\}$$

$$\leq \frac{6(2p^2 + p\gamma_m - 2)}{(p\delta - 2)(p\gamma_m - 2) - p(2+p)}\left(\overline{N}(r, 0; \Im(f^q)) + \overline{N}(r, 0; P[f^q])\right) + S(r, P[f^q])$$
$$+ S(r, f)$$

$$\leq \frac{6(2p^2 + p\gamma_m - 2)}{(p\delta - 2)(p\gamma_m - 2) - p(2+p)}\left\{T(r, \Im(f^q)) + T(r, P[f^q])\right\} + S(r, P[f^q])$$
$$+ S(r, f),$$

which contradicts $n > 5 + \frac{6(2p^2 + p\gamma_m - 2)}{(p\delta - 2)(p\gamma_m - 2) - p(2+p)}$.

Subcase 1.2. Let $p = 1$. We see that

$$(4.7) \quad \overline{N}(r, 1; \mathcal{F}| \geq 2) + \overline{N}_L(r, 1; \mathcal{F}) + \overline{N}_L(r, 1; \mathcal{G}) + \overline{N}_0\left(r, 0; (P[f])'\right)$$

$$\leq \overline{N}(r, 1; \mathcal{G}| \geq 2) + \overline{N}(r, 1; \mathcal{F}| \geq 2) + \overline{N}_0\left(r, 0; (P[f^q])'\right)$$

$$\leq N\left(r, 0; (P[f^q])' \,\Big|\, P[f^q] \neq 0\right) + \frac{1}{2}N\left(r, 0; (\Im(f^q))' \,\Big|\, \Im(f^q) \neq 0\right)$$

$$\quad + S(r, P[f^q]) + S(r, f)$$

$$\leq \overline{N}(r, 0; P[f^q]) + \overline{N}(r, \infty; P[f^q]) + \frac{1}{2}\left\{\overline{N}(r, 0; \Im(f^q)) + \overline{N}(r, \infty; f)\right\}$$

$$\quad + S(r, P[f^q]) + S(r, f).$$

Thus we get from (4.1),

$$(4.8) \quad (n + 1)T(r, \Im(f^q))$$

$$\leq \frac{5}{2}\left\{\overline{N}(r, \infty; f) + \overline{N}(r, 0; \Im(f^q))\right\} + 2\overline{N}\left(r, 0; P[f^q]\right) + \left\{\overline{N}(r, 0; \Im(f^q))\right.$$

$$\left. + \overline{N}\left(r, b; P[f^q]\right) + \overline{N}(r, \infty; f)\right\} + S(r, P[f]) + S(r, f).$$

Similarly for $P[f^q]$, we get

$$(4.9) \quad (n + 1)T(r, P[f^q])$$

$$\leq \frac{5}{2}\left\{\overline{N}(r, \infty; f) + \overline{N}(r, 0; P[f^q])\right\} + 2\overline{N}\left(r, 0; \Im(f^q)\right) + \left\{\overline{N}(r, 0; P[f^q])\right.$$

$$\left. + \overline{N}\left(r, b; \Im(f^q)\right) + \overline{N}(r, \infty; f)\right\} + S(r, P[f^q]) + S(r, f).$$

Adding (4.8) and (4.9), we get

$$(4.10) \quad (n + 1)\left\{T(r, \Im(f^q)) + T(r, P[f^q])\right\}$$

$$\leq 7\overline{N}(r, \infty; f) + \frac{11}{2}\left\{\overline{N}(r, 0; \Im(f^q)) + \overline{N}(r, 0; P[f^q])\right\}$$

$$\quad + \left\{\overline{N}(r, b; \Im(f^q)) + \overline{N}(r, b; P[f^q])\right\} + S(r, P[f^q]) + S(r, f),$$

i.e.,

$$(4.11) \quad \left(n - \frac{11}{2}\right)\left\{T(r, \Im(f^q)) + T(r, P[f^q])\right\}$$

$$\leq 7\overline{N}(r, \infty; f) + S(r, P[f^q]) + S(r, f).$$

So using Lemma 3.6 in (4.11), we get

$$(4.12) \quad \left(n - \frac{11}{2}\right)\left\{T(r, \Im(f^q)) + T(r, P[f^q])\right\}$$

$$\leq \frac{\gamma_m}{(\delta-2)(\gamma_m-2)-3}\left\{\overline{N}(r, 0; \Im(f^q)) + \overline{N}(r, 0; P[f^q])\right\} + S(r, P[f^q])$$
$$+ S(r, f)$$

$$\leq \frac{\gamma_m}{(\delta-2)(\gamma_m-2)-3}\left\{T(r, \Im(f^q)) + T(r, P[f^q])\right\} + S(r, P[f^q]) + S(r, f),$$

which contradicts $n > \frac{11}{2} + \frac{\gamma_m}{(\delta-2)(\gamma_m-2)-3}$.

Subcase 1.3. Let $p = 0$.

Using the Second Fundamental Theorem and Lemma 3.7, we get

$$(4.13) \quad (n+1)\left\{T(r, \Im(f^q)) + T(r, P[f^q])\right\}$$

$$\leq \overline{N}(r, \infty; \Im(f^q)) + \overline{N}(r, \infty; P[f^q]) + \overline{N}(r, 0; \Im(f^q)) + \overline{N}(r, 0; P[f^q])$$
$$+ \overline{N}(r, b; \Im(f^q)) + \overline{N}(r, b; P[f^q]) + \overline{N}(r, 1; F) + \overline{N}(r, 1; G)$$
$$- N_0(r, 0; (\Im(f^q))') - N_0(r, 0; (P[f^q])') + S(r, P[f^q]) + S(r, f)$$

$$\leq 3\overline{N}(r, \infty; f) + 2\overline{N}(r, 0; \Im(f^q)) + 2\overline{N}(r, 0; P[f^q]) + 2\overline{N}(r, b; \Im(f^q))$$
$$+ 2\overline{N}(r, b; P[f^q]) + \overline{N}(r, 1; \mathcal{F}) + \overline{N}(r, 1; \mathcal{G}) - \overline{N}(r, 1; \mathcal{F}| = 1)$$
$$+ \overline{N}_L(r, 1; \mathcal{F}) + \overline{N}_L(r, 1; \mathcal{G}) + S(r, P[f^q]) + S(r, f).$$

Again,

$$\overline{N}(r, 1; \mathcal{F}) + \overline{N}(r, 1; \mathcal{G}) - \overline{N}(r, 1; \mathcal{F}| = 1) \leq \overline{N}_L(r, 1; \mathcal{F}) + N(r, 1; \mathcal{G}),$$

i.e.,

$$\overline{N}(r, 1; \mathcal{F}) + \overline{N}(r, 1; \mathcal{G}) - \overline{N}(r, 1; \mathcal{F}| = 1)$$
$$\leq \frac{1}{2}\left\{\overline{N}_L(r, 1; \mathcal{F}) + \overline{N}_L(r, 1; \mathcal{G}) + N(r, 1; \mathcal{G}) + N(r, 1; \mathcal{F})\right\}.$$

So in view of Lemma 3.4 and Lemma 3.6, from (4.13), we have

$$(4.14) \quad (n+1)\left\{T(r, \Im(f^q)) + T(r, P[f^q])\right\}$$

$$\leq 3\overline{N}(r, \infty; f) + 2\overline{N}(r, 0; \Im(f^q)) + 2\overline{N}(r, 0; P[f^q]) + 2\overline{N}(r, b; \Im(f^q))$$
$$+ 2\overline{N}(r, b; P[f^q]) + \frac{3}{2}\left\{\overline{N}_L(r, 1; \mathcal{F}) + \overline{N}_L(r, 1; \mathcal{G})\right\} + \frac{1}{2}\left\{N(r, 1; \mathcal{F})\right.$$

$$\left. + N(r, 1; \mathcal{G})\right\} + S(r, P[f^q]) + S(r, f),$$

i.e.,

$$(n-6)\left\{T(r,\Im(f^q)) + T\left(r, P[f^q]\right)\right\}$$

$$\leq 6\overline{N}(r,\infty;f) + 3\left\{\overline{N}_L(r,1;\mathcal{F}) + \overline{N}_L(r,1;\mathcal{G})\right\} + S(r, P[f^q]) + S(r,f)$$

$$\leq 6\overline{N}(r,\infty;f) + 3\left\{\overline{N}\left(r,0; P[f^q]\right) + \overline{N}(r,0;\Im(f^q)) + 2\overline{N}(r,\infty;f)\right\}$$

$$+ S(r, P[f^q]) + S(r,f).$$

Thus

$$(n-11)\left\{T(r,\Im(f^q)) + T\left(r, P[f^q]\right)\right\}$$

$$\leq \frac{12\gamma_m}{(\delta-2)(\gamma_m-2)-3}\left\{\overline{N}(r,0;\Im(f^q)) + \overline{N}(r,0; P[f^q])\right\} + S(r, P[f^q])$$

$$+ S(r,f)$$

$$\leq \frac{12\gamma_m}{(\delta-2)(\gamma_m-2)-3}\left\{T(r,\Im(f^q)) + T(r, P[f^q])\right\} + S(r, P[f^q]) + S(r,f),$$

which contradicts $n > 11 + \frac{12\gamma_m}{(\delta-2)(\gamma_m-2)-3}$.

Case 2. Let $\mathcal{H} \equiv 0$.

It is clear that $\mathcal{F}$ and $\mathcal{G}$ share $(1,\infty)$.

By integration twice, we have

$$(4.15) \qquad \mathcal{F} = \frac{\mathcal{A}\mathcal{G} + \mathcal{B}}{\mathcal{C}\mathcal{G} + \mathcal{D}} \quad \text{or} \quad \mathcal{G} = \frac{-\mathcal{D}\mathcal{F} + \mathcal{B}}{\mathcal{C}\mathcal{F} - \mathcal{A}},$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are constants satisfying $\mathcal{A}\mathcal{D} - \mathcal{B}\mathcal{C} \neq 0$.

Next by applying Mokhon'ko's Lemma, we get [9]

$$(4.16) \qquad \begin{aligned} T(r,\Im(f^q)) &= \frac{1}{n}T(r,\mathcal{F}) + S(r,f), \\ T\left(r, P[f^q]\right) &= \frac{1}{n}T(r,\mathcal{G}) + S(r, P[f^q]). \end{aligned}$$

From (4.15), we get $T(r,\mathcal{F}) = T(r,\mathcal{G}) + O(1)$, i.e., $T(r,\Im(f^q)) = T\left(r, P[f^q]\right) + O(1)$. Clearly from equation (4.15), if $\mathcal{C} \neq 0$, we get $\overline{N}(r,\infty;f) = S(r,f)$.

As $\mathcal{A}\mathcal{D} - \mathcal{B}\mathcal{C} \neq 0$, so $\mathcal{A} = \mathcal{C} = 0$ is not possible. So we consider the following cases:

Subcase 2.1. Let $\mathcal{A}\mathcal{C} \neq 0$. This implies $\mathcal{A} \neq 0$ and $\mathcal{C} \neq 0$.

Subcase 2.1.1. Let $\mathcal{B} = 0$. Then we must have $\mathcal{D} \neq 0$ otherwise $\mathcal{A}\mathcal{D} - \mathcal{B}\mathcal{C} = 0$.

In this case, (4.15) reduces to

$$(4.17) \qquad \mathcal{F} = \frac{\mathcal{A}\mathcal{G}}{\mathcal{C}\mathcal{G} + \mathcal{D}}.$$

Since $\overline{N}(r,\infty;f) = S(r,f)$, it follows from (4.17) that $\overline{N}\left(r, -\frac{\mathcal{D}}{\mathcal{C}}; \mathcal{G}\right) = \overline{N}(r,\infty;\mathcal{F}) = S(r,f)$.

Applying Second Fundamental Theorem, we get

$$T(r,\mathcal{G}) \leq \overline{N}(r,\infty;\mathcal{G}) + \overline{N}(r,0;\mathcal{G}) + \overline{N}\left(r, -\frac{\mathcal{D}}{\mathcal{C}};\mathcal{G}\right) + S(r,\mathcal{G})$$

$$\leq \overline{N}(r,\infty;\mathcal{G}) + \overline{N}(r,0;\mathcal{G}) + S(r,P[f^q])$$

$$\leq \overline{N}(r,\infty;f) + \sum_{i=1}^{2}\overline{N}\left(r,\alpha_i;P[f^q]\right) + \overline{N}(r,0;P[f^q]) + S(r,P[f^q])$$

$$\leq \frac{3}{n}T(r,\mathcal{G}) + S(r,P[f^q]),$$

a contradiction as $n > 5$.

Subcase 2.1.2. Let $\mathcal{B} \neq 0$.

In this case, we have

$$\mathcal{G} + \frac{\mathcal{B}}{\mathcal{A}} = \frac{(\mathcal{B}\mathcal{C} - \mathcal{A}\mathcal{D})\mathcal{F}}{\mathcal{A}(\mathcal{C}\mathcal{F} - \mathcal{A})}.$$

By applying Second Fundamental Theorem, we get

$$T(r,\mathcal{G}) \leq \overline{N}(r,\infty;\mathcal{G}) + \overline{N}(r,0;\mathcal{G}) + \overline{N}\left(r, -\frac{\mathcal{B}}{\mathcal{A}};\mathcal{G}\right) + S(r,\mathcal{G})$$

$$\leq \overline{N}(r,\infty;\mathcal{G}) + \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,0;\mathcal{F}) + S(r,P[f^q])$$

$$\leq \overline{N}(r,\infty;f) + \sum_{i=1}^{2}\overline{N}\left(r,\alpha_i;P[f^q]\right) + \overline{N}(r,0;P[f^q]) + \overline{N}(r,0;\Im(f^q))$$

$$+ S(r,P[f^q])$$

$$\leq \frac{4}{n}T(r,\mathcal{G}) + S(r,P[f^q]),$$

which is a contradiction as $n > 5$.

Subcase 2.2. Let $\mathcal{A}\mathcal{C} = 0$.

Subcase 2.2.1. Let $\mathcal{A} = 0$ and $\mathcal{C} \neq 0$.

In this case $\mathcal{B} \neq 0$ and $\mathcal{F} = \frac{1}{\gamma\mathcal{G}+\delta}$, where $\gamma = \frac{\mathcal{C}}{\mathcal{B}}$ and $\delta = \frac{\mathcal{D}}{\mathcal{B}}$.

If $\mathcal{F}$ has no 1-point, then by the Second Fundamental Theorem and (4.16), we get

$$T(r,\mathcal{F})$$

$$\leq \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,1;\mathcal{F}) + S(r,\mathcal{F})$$

$$\leq \overline{N}(r,\infty;f) + \sum_{i=1}^{2}\overline{N}(r,\alpha_i;\Im(f^q)) + \overline{N}(r,0;\Im(f^q)) + S(r,\mathcal{F})$$

$$\leq \frac{3}{n}T(r,\mathcal{F}) + S(r,\mathcal{F}),$$

which contradicts $n > 5$.

Thus $\gamma + \delta = 1$ and $\gamma \neq 0$.

So,

$$\mathcal{F} = \frac{1}{\gamma \mathcal{G} + 1 - \gamma}.$$

From above we get $\overline{N}\left(r, \frac{1}{1-\gamma}; \mathcal{F}\right) = \overline{N}(r, 0; \mathcal{G})$.

If $\gamma \neq 1$, by the Second Fundamental Theorem and (4.16), we get

$$T(r, \mathcal{F}) \leq \overline{N}(r, \infty; \mathcal{F}) + \overline{N}(r, 0; \mathcal{F}) + \overline{N}\left(r, \frac{1}{1-\gamma}; \mathcal{F}\right) + S(r, \mathcal{F})$$

$$\leq \overline{N}(r, \infty; \Im(f^q)) + \sum_{i=1}^{2} \overline{N}(r, \alpha_i; \Im(f^q)) + \overline{N}(r, 0; \Im(f^q))$$

$$+ \overline{N}(r, 0; P[f^q]) + S(r, \mathcal{F})$$

$$\leq \frac{4}{n} T(r, \mathcal{F}) + S(r, \mathcal{F}),$$

which is a contradiction as $n > 5$.

Thus $\gamma = 1$ and hence $\mathcal{F}\mathcal{G} \equiv 1$ which yields

$$a^2 (\Im(f^q))^n (P[f^q])^n = n^2(n-1)^2 \prod_{i=1}^{2} (\Im(f^q) - \alpha_i) \prod_{i=1}^{2} (P[f^q] - \alpha_i).$$

Let z_1 be a pole of f of order r, then it must be a pole of $\Im(f^q)$ and $P[f^q]$ of order $\overline{d}(P)r$ and $\overline{d}(P)r + \Gamma_P$ respectively and from above we get $n\overline{d}(P)r + n(\overline{d}(P)r + \Gamma_P) = 2\overline{d}(P)r + 2(\overline{d}(P)r + \Gamma_P)$, i.e., $n = 2$. This is not possible since from the hypothesis of Theorem 2.1 we see that $n > 2$. Thus from the above equation it is clear that f has no pole.

Let $\Im(f^q) - \alpha_i = b_1 \prod_{j=1}^{r_i} (f - \alpha_{ij})^{p_{ij}}$, where $1 \leq i \leq 2, 1 \leq r_i \leq \overline{d}(P)$ and $1 \leq p_{ij} \leq \overline{d}(P)$, r_i, $p_{ij} \in \mathbb{N}$. Let z_0 be a α_{ij}-point of f of order s, $j = 1, 2, \ldots, r_i$, then as these types of points can only be neutralized by the zeros of $P[f]$, we must have $s \geq q(\geq k+1)$. Consequently we have $\overline{N}(r, \alpha_{ij}; f) \leq \frac{1}{k+1} N(r, \alpha_{ij}; f)$, for $j = 1, 2, \ldots, r$ for $i = 1, 2$. Thus by the Second Fundamental Theorem, we get

$$(4.18) \qquad (r_1 + r_2 - 1)T(r, f) \leq \left(\sum_{i=1}^{2} r_i - 1\right) T(r, f)$$

$$\leq \overline{N}(r, \infty; f) + \sum_{j=1}^{r_1} \overline{N}(r, \alpha_{1j}; f) + \sum_{j=1}^{r_2} \overline{N}(r, \alpha_{2j}; f) + S(r, f)$$

$$\leq \sum_{j=1}^{r_1} \frac{1}{k+1} N(r, \alpha_{1j}; f) + \sum_{j=1}^{r_2} \frac{1}{k+1} N(r, \alpha_{2j}; f) + S(r, f)$$

$$\leq \frac{(r_1 + r_2)}{k+1} T(r, f) + S(r, f),$$

which is a contradiction if $r_1 + r_2 \geq 3$.

Next suppose $r_1 + r_2 = 2$, which implies that $r_1 = 1 = r_2$. So let $\Im(f^q) - \alpha_1 = b_1(f - \alpha_1^*)^{\overline{d}(P)}$ and $\Im(f^q) - \alpha_2 = b_1(f - \alpha_2^*)^{\overline{d}(P)}$. Now by the same argument as above if we assume z_1 be a α_1^*-point of f of order s, we see that $n \leq \overline{d}(P)s$ and hence similar to (4.18), we get, $T(r, f) \leq \frac{2\overline{d}(P)}{n} T(r, f) + S(r, f)$, which is a contradiction as $n > 2\overline{d}(P)$.

Subcase 2.2.2. Let $\mathcal{A} \neq 0$ and $\mathcal{C} = 0$.

In this case $\mathcal{D} \neq 0$ and $\mathcal{F} = \lambda \mathcal{G} + \mu$, where $\lambda = \frac{\mathcal{A}}{\mathcal{D}}$ and $\mu = \frac{\mathcal{B}}{\mathcal{D}}$.

If $\mathcal{F}$ has no 1-point, then similarly as above we get a contradiction.

Thus $\lambda + \mu = 1$ with $\lambda \neq 0$.

Clearly $\overline{N}\left(r, 0; \mathcal{G} + \frac{1-\lambda}{\lambda}\right) = \overline{N}(r, 0; \mathcal{F})$.

If $\lambda \neq 1$, then by the Second Fundamental Theorem and (4.16), we get

$$T(r, \mathcal{G}) \leq \overline{N}(r, \infty; \mathcal{G}) + \overline{N}(r, 0; \mathcal{G}) + \overline{N}\left(r, 0; \mathcal{G} + \frac{1-\lambda}{\lambda}\right) + S(r, \mathcal{G})$$

$$\leq \overline{N}(r, \infty; P[f^q]) + \sum_{i=1}^{2} \overline{N}\left(r, \alpha_i; P[f^q]\right) + \overline{N}\left(r, 0; P[f^q]\right)$$

$$+ \overline{N}(r, 0; \Im(f^q)) + S(r, P[f^q])$$

$$\leq \frac{5}{n} T(r, \mathcal{G}) + S(r, \mathcal{G}),$$

which contradicts $n > 5$.

Thus $\lambda = 1$ and hence $\mathcal{F} \equiv \mathcal{G}$. Therefore

$$n(n-1)\Im(f^q)^2 P[f^q]^2 \left\{ (\Im(f^q))^{n-2} - (P[f^q])^{n-2} \right\}$$

$$-2n(n-2)b \, \Im(f^q)P[f^q]\left\{ (\Im(f^q))^{n-1} - (P[f^q])^{n-1} \right\}$$

$$+(n-1)(n-2)b^2 \left\{ (\Im(f^q))^n - (P[f^q])^n \right\} = 0.$$

By substituting $h = \frac{P[f^q]}{\Im(f^q)}$, we get

(4.19) $\quad n(n-1)(\Im(f^q))^2 h^2 \left(h^{n-2} - 1\right) - 2n(n-2)bh(\Im(f^q)) \left(h^{n-1} - 1\right)$

$\qquad + (n-1)(n-2)b^2 \left(h^n - 1\right) = 0.$

If h is non constant, then using Lemma 3.8, we obtained from (4.19),

$$\left\{ n(n-1)h \, \Im(f^q)(h^{n-2} - 1) - n(n-2)b(h^{n-1} - 1) \right\}^2$$

$$= -n(n-2)b^2(h-1)^4 \prod_{i=1}^{2n-6} (h - \beta_i).$$

By applying the Second Fundamental Theorem, we get

$$(2n-6)T(r, h) \leq \overline{N}(r, \infty; h) + \overline{N}(r, 0; h) + \sum_{i=1}^{2n-6} \overline{N}(r, \beta_i; h) + S(r, h)$$

$$\leq \overline{N}(r,\infty;h) + \overline{N}(r,0;h) + \frac{1}{2}\sum_{i=1}^{2n-6} N(r,\beta_i;h) + S(r,h)$$

$$\leq (n-1)T(r,h) + S(r,h),$$

which contradicts $n > 5$.

Thus h is constant. Hence as f is non-constant and $b \neq 0$, we get from the equation (4.19), that $h^{n-2}-1 = 0$, $h^{n-1}-1 = 0$ and $h^n-1 = 0$, i.e., $h^d-1 = 0$, where $d = \gcd(n, n-1, n-2) = 1$. Consequently $\Im(f^q) \equiv P[f^q]$.

An elementary calculation shows that

$$f(z) = c \, \exp\left(\frac{\lambda}{q}z\right),$$

is a certain solution of $\Im(f^q) \equiv P[f^q]$, where c is a non-zero constant with $\lambda^{Q_{M_j}} = 1$ for all $j = 1, 2, \ldots, t$.   □

Proof of Theorem 2.2. We suppose that f be a non-constant entire function. So, $\overline{N}(r,\infty;f) = S(r,f)$. The rest of the proof can be carried out exactly same way as in the line of the proof of Theorem 2.1.   □

Proof of Corollary 2.1. Let $\mathcal{H} \not\equiv 0$. Let f be a non-constant entire function such that $N(r,0;f) = S(r,f)$, then it is clear that $N(r,0;f^q) = S(r,f)$. So from Lemma 3.3, it follows that $\overline{N}(r,0;(f^q)^{(k)}) = S(r,f)$. Here we put $\Im(f^q) = f^q$ and $P[f^q] = (f^q)^{(k)}$.

In this situation proceeding exactly in the same way as done in Theorem 2.1, we get from (4.5), $n > 2$, from (4.10), $n > 2$ and from (4.14), $n > 4$. So we omit the details.   □

5. Concluding remarks and some open questions

Based on our discussions in Note 1.3, and also in Theorems 2.1 and 2.2, we observe that it is not always possible to find the general meromorphic solution, rather than a particular solution, of the relation $\Im(f^q) \equiv P[f^q]$.

So for the future investigations in this direction, we now pose the following questions.

Question 5.1. What is the general meromorphic solution of $\Im(f^q) \equiv P[f^q]$?

Question 5.2. Keeping all other conditions intact of Theorems 2.1 and 2.2, is it possible to extend the expression $\Im(f^q)$ up to another differential polynomial $Q[f^q]$ to get the same conclusions?

Question 5.3. Without imposing any extra conditions, it possible to reduce the cardinality of the set $\mathcal{S}$ further in Theorems 2.1 and 2.2, to get the same conclusions?

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