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Results in Mathematics

An Investigation on the Conjecture of Chen and Yi

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Abstract. In the paper, we have investigated on a conjecture posed by Chen and Yi (Results Math 63:557–565, 2013) concerning the uniqueness problem of meromorphic functions f sharing three distinct values with their difference $\mathcal{L}_c(f)$. We have proved the conjecture for finite ordered meromorphic functions. Some examples have been exhibited in the paper to show that the main result is true also for the meromorphic function of infinite order, but we are unable to prove our results for the function of infinite order, and hence we conjecture it. The main results in the paper also generalized a result of Zhang and Liao (Sci China Math 57(10):2143–2152, 2014). This research also shows that when a meromorphi function f satisfies a certain relation of the type $\mathcal{L}_c(f) \equiv f$, then it can be found the class of all such functions.

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1. Introduction, Definitions and Results

In this paper, the term "meromorphic (resp. entire)" will always mean meromorphic (resp. entire) in the whole complex plane $\mathbb{C}$. Meromorphic functions are always non-constant, unless specifically stated otherwise. We shall adopt the standard notations of the Nevanlinna's value distribution theory of meromorphic functions such as $T(r, f)$, $m(r, f)$, and $N(r, f)$ ($\overline{N}(r, f)$) from [10, 22]. Throughout the paper, we denote $\mathbb{C}^*$ by $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

We say that two meromorphic functions f and g share the value a **CM** (resp., **IM**), whenever $f - a$ and $g - a$ have the same set of zeros with the same multiplicities (resp. ignoring multiplicities). Also we say that f and g share the value ∞ CM(resp., **IM**) whenever $1/f$ and $1/g$ share the value 0 $\mathbb{CM}$ (resp., **IM**).

It is well known that a pair of meromorphic functions f and g would be identically equal to each other if f and g share five distinct values **IM**. This is the famous *Nevanlinna's Five-value Theorem.* Also the *Nevanlinna's Four-value Theorem* states that if a pair of meromorphic functions f and g share four value **CM**, then f is a bilinear transformation of g. The beauty of these results lies on the fact that it has no counter part in the real function theory. The study will be more interesting if the second function g is related with f.

The condition "f and g share four values **CM**" has been weakened to "f and g share two values **CM** and two values **IM**" by Gundersen [6,7], as well as Mues [16]. But whether the condition can be weakened to "f and g share three values **IM** and one value **CM**" is still open.

In 1976, Rubel and Yang [18], investigated on the uniqueness of the entire function when sharing two values a, b with its derivative by proving the following classical result.

Theorem A [18]. *Let f be a non-constant entire function. If f and f' share two distinct finite values a, b CM, then $f \equiv f'$.*

It was Brück [2] who investigated further on the entire function sharing one value with its first derivative and posed the following conjecture.

Conjecture 1.1 [2]. *Let f be a non-constant entire function satisfying the super order $\rho_2(f) < \infty$, being not a positive integer. If f and f' share one finite value a CM, then $f' - a \equiv c\,(f - a)$ holds for some constants $c \neq 0$.*

We use the standard notations of the Nevanlinnas value distribution theory of meromorphic functions (see [10,22]). In addition with this, we use the notation $\rho(f)$ to denote the order growth of f. Finally, $\rho_2(f)$ denoted the hyper-order (see [20]) of f which is defined by

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

Recently, the difference analogue of the lemma on the logarithmic derivative, and hence Nevanlinna theory for the differences of a meromorphic function f have been studied by many researchers [1,3,8,9,11–13] focusing on the uniqueness problem of meromorphic functions sharing some values with their difference operators or shifts. Now-a-days investigating on the problems of sharing values by a meromorphic functions with its differences or shift become an extensive subfield of the modern uniqueness theory. It is well known that $\Delta_c f$ can be considered as the difference counterpart of f' in Theorem A. In this direction, we mention an interesting result here.

Theorem B [4]. *Let f be a finite order transcendental entire function which has a finite Borel exceptional value a, and let $c \in \mathbb{C}^*$ be a constant such that*

$f(z+c) \not\equiv f(z)$. If $\Delta_c f(z) = f(z+c) - f(z)$ and $f(z)$ share the value a ***CM****, then $a = 0$, and*

$$\Delta_c f \equiv \mathcal{A}\, f(z),$$

where $\mathcal{A}$ is a non-zero constant.

Remark 1.1. Clearly Theorem B shows that if f has a non-zero finite Borel exceptional value a, then for $c \neq 0$, the value a is not shared by $\Delta_c f$ and $f(z)$.

Following example clears this fact.

Example 1.1. Let $f(z) = be^z + a$ has the Borel exceptional value a. Clearly, for any $c \neq 2k\pi i$, $k \in \mathbb{Z}$, the vale a is not shared by $f(z+c) - f(z) = b(e^c - 1)e^z$ and $f(z)$.

In this direction, one natural question arises as follows.

Question 1.1. Is it possible to omit the condition "f has a finite Borel exceptional value" in Theorem B?

The following result, which answers the above question affirmatively, and is also similar to the assumption in *Brück* conjecture, in which the order of growth $\rho(f)$ is not an integer or infinite.

Theorem C [4]. *Let f be a transcendental meromorphic function such that its order of growth $\rho(f)$ is not an integer or infinite, and $c \in \mathbb{C}^*$ such that $\Delta_c f \not\equiv 0$. If $\Delta_c f$ and $f(z)$ share three distinct values a, b, ∞* ***CM****, then $\Delta_c f \equiv f(z)$.*

Recently Zhang and Liao [23] proved a result by showing that the above conjecture is true when f is an entire function and Theorem A is still valid when f' is replaced by $\Delta_c f$. Following is the result.

Theorem D [23]. *Let $f(z)$ be a transcendental entire function of finite order, and a, b be two distinct constants. If $\Delta_1 f(\not\equiv 0)$ and f share a, b* ***CM****, then $\Delta_1 f \equiv f$. Furthermore, $f(z)$ must be of the following form $f(z) = 2^z h(z)$, where $h(z)$ is a periodic entire function with period* 1.

In [23], authors gave the following example to show that in Theorem D, sharing two distinct values **CM** can not be relaxed to sharing one value **CM**.

Example 1.2. Let $f(z) = e^{\pi i z}$, one can check that f and $\Delta_c f$ share 0 **CM** but $f \not\equiv \Delta_c f$.

The next two examples show that, in Theorem D, it is not necessary that the function has to be of finite order.

Example 1.3. Let $f(z) = 2^{z} e^{c\sin(2\pi z)+d}$, where $c(\neq 0)$, $d \in \mathbb{C}$. Clearly $\Delta_1 f$ and f share any two distinct values a and b **CM**, and also we see that $\Delta_1 f \equiv f$. Note that f has the form $f(z) = 2^{z} h(z)$, where $h(z) = e^{c\sin(2\pi z)+d}$ is a periodic entire function with period 1.

Example 1.4. Let $f(z) = 2^z \dfrac{e^{2\pi iz}-1}{e^{c\sin(2\pi z)+d}}$, where $c(\neq 0)$, $d \in \mathbb{C}$. Clearly $\Delta_1 f$ and f share any two distinct values a and b **CM**, and also we see that $\Delta_1 f \equiv f$. Note that f has the form $f(z) = 2^z h(z)$, where $h(z) = \dfrac{e^{2\pi iz}-1}{e^{c\sin(2\pi z)+d}}$ is a periodic entire function with period 1.

For the generalization of $\Delta_c f = f(z+c) - f(z)$ further, we now define the difference operator of an entire (meromorphic) function f as $\mathcal{L}_c(f) := c_1 f(z+c) + c_0 f(z)$, where $c_0, c_1 \in \mathbb{C}^*$. Clearly for the particular choice of the constants $c_0 = -1$ and $c_1 = 1$, we get back $\mathcal{L}_c f = \Delta_c f$.

Since no attempts, till now, have so far been made by any researchers investigating on the uniqueness problem when an entire function f and its difference $\mathcal{L}_c(f)$ sharing a, b **CM**, and stressing to find the class of all such entire functions, satisfying the relation $\mathcal{L}_c(f) \equiv f$.

Therefore, in this paper, we are mainly interested in the further generalization of Theorem D by replacing $\Delta_1 f$ by a more general setting $\mathcal{L}_c(f)$. We have also stressed to find the class of all the functions satisfying the relation $\mathcal{L}_c(f) \equiv f$.

Following is a main result in this paper.

Theorem 1.1. *Let f be a transcendental entire function of finite order, and a, b be two distinct constants. If $\mathcal{L}_c(f)(\not\equiv 0)$ and f share a, b **CM**, then $\mathcal{L}_c(f) \equiv f$. Furthermore, $f(z)$ must be of the following form*

$$f(z) = \begin{cases} \left(\dfrac{1-c_0}{c_1}\right)^{\frac{z}{c}} h(z), & \text{when } c_0 + c_1 \neq 1 \\ h(z), & \text{when } c_0 + c_1 = 1, \end{cases}$$

where $h(z)$ is a periodic entire function with period c.

Remark 1.2. One can see that when $\mathcal{L}_c(f) \equiv f$ with the condition $c_1 + c_0 = 1$, then the function f is a periodic with period c.

Towards the improvement of the Theorem D, one may ask the following natural question.

Question 1.2. Is Theorem D true for transcendental meromorphic function also?

Followings are two supportive examples towards the answer of the Question 1.2 affirmatively.

Example 1.5. Let $f(z) = 2^{\frac{z}{c}} \frac{\sin\left(\frac{2\pi z}{c}\right)}{\cos\left(\frac{2\pi z}{c}\right)-1}$, where $c \in \mathbb{C}^*$. Clearly $\Delta_c f$ and f share any two distinct complex numbers a and b **CM** satisfying $\Delta_c f \equiv f$.

Example 1.6. Let $f(z) = 2^{\frac{z}{c}} \frac{a+b}{\exp\left(\frac{2\pi iz}{c}\right)+a+b}$, where $c \in \mathbb{C}^*$. Clearly $\Delta_c f$ and f share any two distinct complex numbers a and b **CM** satisfying $\Delta_c f \equiv f$.

From the above two examples we see that, if one considers a meromorphic function instead of entire, then to get the desired relation, the functions f and $\Delta_c f(\not\equiv 0)$ must have to share ∞ **CM**.

In [4], Chen and Yi conjectured that the conclusion of Theorem C still holds if the restriction of $\rho(f)$ in Theorem C is omitted. An worth noticing fact is that Theorem D shows that the conjecture is correct if f is an entire function of finite order. Now-a-days further investigation on this conjecture is going on, and many researchers are engaged to solve the conjecture.

Recently, Lü and Lü [14] studied the conjecture, and proved it holds for the meromorphic functions of finite order also, and obtained the following result.

Theorem E [14]. *Let f be a transcendental meromorphic function of finite order, and let $\Delta_c(f) = f(z+c) - f(z)$ $(\not\equiv 0)$, where $c \neq 0$ is a finite number. If $\Delta_c f$ and f share three distinct values a, b, ∞ **CM**, then $f \equiv \Delta_c f$.*

Remark 1.3. We see that Examples 1.5 and 1.6 are the supportive examples for the validity of Theorem E.

So we want to investigate further Theorem E for a meromorphic functions f and its difference $\mathcal{L}_c(f)$.

Therefore, relevant to this investigation, our aim is to study Theorems C, D and E, replacing $\Delta_c f$ by $\mathcal{L}_c(f)$ and also to find the class of the functions. Apart from this, since our investigation is on the uniqueness problem between a meromorphic function $f(z)$ and its shift $f(z+c)$ or differences $\Delta_c f$ or $\mathcal{L}_c(f)$, so it is quite natural to investigate on the periodicity of the function $f(z)$ under some suitable condition. Clearly, when $\Delta_c f \equiv f$, the function could not be a periodic function.

Following is another result in the paper which improved Theorems C, D and E.

Theorem 1.2. *Let f be a transcendental meromorphic function of finite order, and a, b be two distinct constants. If $\mathcal{L}_c(f)\left(\not\equiv \frac{d_1e^{\alpha}+d_2}{d_3e^{\beta}+d_4}\right)$ where $d_j \in \mathbb{C}$ and α, β are polynomials in z, and f share a, b and ∞ **CM**, then $\mathcal{L}_c(f) \equiv f$. Furthermore, $f(z)$ must be of the following form*

$$f(z) = \begin{cases} \left(\dfrac{1-c_0}{c_1}\right)^{\frac{z}{c}} g(z), & \text{when } c_0 + c_1 \neq 1 \\ g(z), & \text{when } c_0 + c_1 = 1, \end{cases}$$

where $g(z)$ is a periodic meromorphic function with period c.

The following examples ensure about the existence of functions which satisfy the conditions of Theorem 1.2.

Example 1.7. Let $f(z)=\frac{e^z}{e^{\frac{2\pi iz}{\log\left(\frac{1-c_0}{c_1}\right)}}-1}$, where $c_1, c_2\in\mathbb{C}^*$ with $c_0+c_1\neq 1$. Clearly f is a transcendental meromorphic function (with $\rho(f)=1<\infty$) which shares any a, b and ∞ CM, and hence $\mathcal{L}_c(f)\equiv f$, when $e^c=\frac{1-c_0}{c_1}$.

Example 1.8. Let $f(z)=\frac{e^z\, e^{\sin\left(\frac{2\pi z}{\log\left(\frac{1-c_0}{c_1}\right)}\right)}}{e^{\frac{2\pi iz}{\log\left(\frac{1-c_0}{c_1}\right)}}-1}$, where $c_1, c_2\in\mathbb{C}^*$ with $c_0+c_1\neq 1$. Clearly f is a transcendental meromorphic function (with $\rho(f)=\infty$) which shares any a, b and ∞ **CM**, and hence $\mathcal{L}_c(f)\equiv f$, when $e^c=\frac{1-c_0}{c_1}$.

From the Examples 1.7 and 1.8, one can see that the assumptions "the order of growth $\rho(f)$ is finite" in Theorem 1.2 and "the order of growth $\rho(f)$ is not an integer or infinite" in Theorem C, can be omitted.

Remark 1.4. We observe that in Theorem 1.2 when $c_0+c_1=1$ with $\mathcal{L}_c(f)\equiv f$, the function $f(z)$ would be a c-periodic meromorphic function.

Following are some supportive example of Remark 1.4.

Example 1.9. Let $f(z)=\frac{e^{\sin z}+e^{-\sin z}}{e^{2iz}-1}$. Then $\mathcal{L}_\pi(f)=f(z)$, when $c_0+c_1=1$.

Example 1.10. For $c\in\mathbb{C}^*$, suppose $f(z)=\frac{e^{2iz}\cos\left(\frac{2z\pi}{c}\right)}{\sin\left(\frac{2z\pi}{c}\right)-\cos\left(\frac{2z\pi}{c}\right)}$. Then $\mathcal{L}_c(f)$ $=f(z)$, when $c_0+c_1=1$.

Example 1.11. Let $f(z)=\frac{a\, e^{2iz}+b\,\tan(z)}{c\,\tan(z)+d}$, where $a, b, c, d\in\mathbb{C}^*$. Then $\mathcal{L}_\pi(f)$ $=f(z)$, when $c_0+c_1=1$.

The next example shows that **CM** sharing can not be relaxed to IM sharing in our main result.

Example 1.12. Let $f(z)=ae^{\sin z}$, where $a\in\mathbb{C}^*$ and we choose the constants $c_0=\frac{1}{2}=c_1$. One can check that f and $\mathcal{L}_\pi(f)=a\frac{\left(e^{\sin z}\right)^2+1}{2e^{\sin z}}$ share the values $-a$, a and ∞ IM, and we see that neither $\mathcal{L}_\pi(f)\equiv f$ nor f has the specific form.

We have the following corollary.

Corollary 1.1. *Let f be a transcendental meromorphic function of finite order, and a, b be two distinct constants. If $\Delta_c f(\not\equiv 0)$ and f share a, b and ∞ **CM**, then $\Delta_c f\equiv f$. Furthermore, $f(z)$ must be of the following form $f(z)=2^{\frac{z}{c}}h(z)$, where $h(z)$ is a periodic meromorphic function with period c.*

2. Some Lemmas

In this section, we are going to discuss some lemmas, which we will use frequently to prove our main result.

Lemma 2.1 [5]. *Let f be a meromorphic function with a finite order ρ, and c be a non-zero constant. Then for any $\epsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = O\left(r^{\rho-1+\epsilon}\right). \tag{2.1}$$

The Eq. (2.1) in which ρ is the (finite) order of f, and $\epsilon > 0$, implies

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f),$$

possibly outside a set of finite logarithmic measure.

Lemma 2.2 [15,19]. *If $\mathcal{R}(f)$ is rational in f and has small meromorphic coefficients, then*

$$T(r, \mathcal{R}(f)) = \deg_f(\mathcal{R})T(r, f) + S(r, f).$$

Lemma 2.3 [5]. *Let f be a transcendental meromorphic function with finite order ρ, and c be a non-zero constant. Then for each $\epsilon > 0$, we have*

$$\begin{aligned} &T(r, f(z+c)) = T(r, f) + O\left(r^{\rho-1+\epsilon}\right) + O\left(\log r\right), \quad \textit{i.e.,} \\ &T(r, f(z+c)) = T(r, f) + S(r, f), \end{aligned}$$

possibly out side of a finite logarithmic measure.

Lemma 2.4 [21]. *Suppose that $f_1, f_2, \ldots, f_n$ $(n \geqslant 2)$ be meromorphic functions and $g_1, g_2, \ldots, g_n$ be entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;

(ii) $g_j - g_k$ *are not constants for* $1 \leqslant j < k \leqslant n$;

(iii) *for* $1 \leqslant j \leqslant n$, $1 \leqslant l < k \leqslant n$, $T(r, f_j) = 0\{T\left(r, e^{g_l - g_k}\right)\}$ $(r \to +\infty,\ r \notin E)$.

Then $f_j \equiv 0$, *for* $j = 1, 2, \ldots, n$.

Lemma 2.5. *Let f be a non-constant meromorphic function such that f and $\mathcal{L}_c(f) = c_1 f(z+c) + c_0 f(z)$ share a, b and ∞* ***CM*** *where c, c_0, $c_1(\neq 0) \in \mathbb{C}^*$, then f is not a rational function.*

Proof. Our proof will be based on the method of contradiction. Let if possible f be a rational function. Then $f(z) = \frac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are two polynomials relatively prime to each other and $P(z)Q(z) \not\equiv 0$. We now define the sets $E(0, P) = \{z : P(z) = 0\}$ and $E(0, Q) = \{z : Q(z) = 0\}$.

Thus we have

$$E(0,P)\cap E(0,Q)=\phi \tag{2.2}$$

Therefore

$$\begin{aligned} c_1f(z+c)+c_0f(z) &= c_1\frac{P(z+c)}{Q(z+c)}+c_0\frac{P(z)}{Q(z)} \\ &= \frac{c_1P(z+c)Q(z)+c_0P(z)Q(z+c)}{Q(z+c)Q(z)} \\ &= \frac{P_1(z)}{Q_1(z)}, \text{(say)} \end{aligned}$$

where $P_1(z)$ and $Q_1(z)$ are two relatively prime polynomials and $P_1(z)Q_1(z) \not\equiv 0$.

Again since $E(a,f)=E(a,c_1f(z+c)+c_0f(z))$ and f is a rational function, then there must exists a polynomial $h(z)$ such that

$$c_1f(z+c)+c_0f(z)-a=(f-a)h(z).$$

i.e.,

$$\frac{c_1P(z+c)Q(z)+c_0P(z)Q(z+c)}{Q(z+c)Q(z)}-a\equiv\left(\frac{P(z)}{Q(z)}-a\right)h(z). \tag{2.3}$$

Case 1. Let $P(z)$ is non-constant.

Then by the *Fundamental Theorem of Algebra*, there exists $z_0\in\mathbb{C}$ such that $P(z_0)=0$. Then from (2.3), we get

$$c_1\frac{P(z_0+c)}{Q(z_0+c)}\equiv(1-h(z_0))a. \tag{2.4}$$

Subcase 1.1. Let $a=0$.

Then from (2.4), we see that $P(z_0+c)=0$. Then we can deduce from (2.2) that $P(z_0+mc)=0$ for all positive integer m. However, this is impossible, and hence we conclude that the polynomial $P(z)$ is a non-zero constant.

Subcase 1.2. Let $a\neq 0$. Then from (2.4), we get

$$P(z_0+c)\equiv\frac{a}{c_1}(1-h(z_0))Q(z_0+c).$$

Next proceeding exactly same way as done in above, we can deduce that

$$P(z_0+mc)\equiv\frac{a}{c_1}(1-h(z_0))Q(z_0+mc). \tag{2.5}$$

From (2.4) and (2.5), we see that

$$\frac{P(z_0+c)}{Q(z_0+c)}=\frac{P(z_0+mc)}{Q(z_0+mc)},\quad \text{for all positive integers } m,$$

which contradicts the fact that $E(0,P)\cap E(0,Q)=\phi$.

Therefore, we see that $f(z)$ takes the form $f(z)=\dfrac{d}{Q(z)}$, where $P(z)=d=\text{constant }(\neq 0)$.

Case 2. Let $Q(z)$ be non-zero constant.

Now

$$c_1 f(z+c) + c_0 f(z) = \frac{c_1 d\, Q(z) + c_0 d\, Q(z+c)}{Q(z+c)Q(z)}. \tag{2.6}$$

Since $E(b,f) = E(b, c_1 f(z+c) + c_0 f(z))$, then there exists a polynomial $h_1(z)$ such that $c_1 f(z+c) + c_0 f(z) - b = (f-b)h_1(z)$. i.e.,

$$c_1\, Q(z) + c_0 Q(z+c) \equiv \frac{d - bQ(z)}{d} h_1(z) Q(z+c). \tag{2.7}$$

Since $Q(z)$ and hence $Q(z+c)$ be non-constant polynomials, so by the *Fundamental Theorem of Algebra*, there exist z_0 and z_1 such that $Q(z_0) = 0 = Q(z_1+c)$.

Subcase 2.1. When $Q(z_0) = 0$, then from (2.7), we see that $h_1(z_0) = -\dfrac{c_0}{d}$, which is absurd.

Subcase 2.2. When $Q(z_1+c) = 0$, then from (2.7), we get $Q(z_1) = 0$, which is not possible.

Thus we conclude that $Q(z)$ is a non-zero constant, say d_2. So we have $f(z) = \dfrac{d}{d_2}$, a constant, which is a contradiction.

This completes the proof. □

Lemma 2.6. *Let f be an entire function and $\mathcal{L}_c(f) = c_1 f(z+c) + c_0 f(z)$ be such that*

$$\frac{\mathcal{L}_c(f) - a}{f - a} \equiv e^{\alpha} \text{ and } \frac{\mathcal{L}_c(f) - b}{f - b} \equiv e^{\beta}, \tag{2.8}$$

where $ab \neq 0$, and $\alpha(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0$ and $\beta(z) = \beta_m z^m + \beta_{m-1} z^{m-1} + \cdots + \beta_1 z + \beta_0$, $\alpha_n (\neq 0), \ldots, \alpha_0$ and $\beta_m (\neq 0), \ldots, \beta_0 \in \mathbb{C}$, then $n = m$ and $|\alpha_n| = |\beta_m|$.

Proof. Without any loss of generality, we may assume that $a = 1$, and set $g := f - a$. Then (2.8) becomes

$$\mathcal{L}_c(g) - 1 = g(z) e^{\alpha(z)} - (c_0 + c_1). \tag{2.9}$$

Differentiating both sides of (2.9), we get

$$e^{\alpha(z)} = \frac{c_1 g'(z+c) + c_0 g'(z)}{g(z)\alpha'(z) + g'(z)}. \tag{2.10}$$

Also it follows from (2.9) that

$$e^{\alpha(z)} = \frac{c_1 g(z+c) + c_0 g(z) + \mathcal{D}}{g(z)}, \tag{2.11}$$

where $\mathcal{D} = c_1 + c_0 - 1$.

By combining (2.10) and (2.11), it follows that

$$\mathcal{G}(z) g(z) - \mathcal{D}\left(\alpha'(z) + \frac{g'(z)}{g(z)}\right) \equiv 0, \tag{2.12}$$

where $\mathcal{G}(z)$ can be found as follows

$$\mathcal{G}(z) = \frac{c_1 g'(z+c)}{g(z)} - \frac{c_1 g'(z)g(z+c)}{g^2(z)} - \frac{c_1\alpha'(z)g(z+c)}{g(z)} - c_0\alpha(z).$$

If we choose $c_1+c_0 = 1$ i.e., $\mathcal{D} = 0$, then from (2.12) we have $\mathcal{G}(z)g(z) = 0$, and this implies that $\mathcal{G}(z) = 0$ or $g(z) = 0$. If $g(z) = 0$, then one can have $f(z) = a$, a constant, which is possible. So we must have $\mathcal{G}(z) = 0$.

Now on simplification, we obtained

$$c_1\Big[g(z)g'(z+c) - g'(z)g(z+c)\Big] = g(z)\Big[c_1\alpha'(z)g(z+c) + c_0 g(z)\alpha(z)\Big]. \tag{2.13}$$

Since $c_1 + c_0 = 1$, so from (2.9), we see that

$$e^{\alpha(z)} = \frac{\mathcal{L}_c(g)}{g} \quad \text{i.e.,} \quad g(z+c) = \frac{1}{c_1}\left(e^{\alpha(z)} - c_0\right) g(z). \tag{2.14}$$

It is clear from (2.14) that $\mathcal{L}_c(g)$ and g share the value 0 CM. One can also check that if $g(z)$ has a zero at $z = z_0$ (say), then $g(z+c)$ must have the same zero at $z = z_0$, otherwise, we see from (2.14) that $e^{\alpha(z)}$ would have pole at $z = z_0$, which is a contradiction.

Regarding the common zeros of $g(z)$ and $g(z+c)$, we have only three possible cases.

(a) When $g(z)$ is periodic i.e., $g(z+c) = g(z)$ for all $z \in \mathbb{C}$.
(b) When $g(z)$ is non-periodic but $g(z)$ and $g(z+c)$ have common zeros with same multiplicities.
(c) When $g(z)$ is non-periodic but $g(z)$ and $g(z+c)$ have common zeros with different multiplicities

Case a. Suppose that $g(z)$ is periodic. Then, we have $g(z+c) = g(z)$ and hence $g'(z+c) = g'(z)$. With this, we see that $g(z)g'(z+c) - g'(z)g(z+c) = 0$, and since $g(z) \not\equiv 0$, hence it follows from (2.13) that

$$c_1\alpha'(z) + c_0\alpha(z) = 0. \tag{2.15}$$

The non-trivial solution of the Eq. (2.15) is $\alpha(z) = k\ e^{-\frac{c_0}{c_1}z}$, where k is some non-zero complex number. This contradicts the fact that $\alpha(z)$ is a polynomial in z. Again, $\alpha(z)$ being a solution of (2.15), has to be 0, which again contradicts the assumption that $\alpha(z)$ is non-zero.

Case b. Let $g(z)$ is non-periodic but $g(z)$ and $g(z+c)$ have zeros with same multiplicities. Let z_0 be a common zero of $g(z)$ and $g(z+c)$ of same multiplicity $p \geqslant 1$.

First we claim that z_0 can not be a zero of the polynomial $\alpha(z)$. On contrary, let z_0 be a zero of $\alpha(z)$ of multiplicity $q(\geqslant 1)$. It follows from (2.13) that z_0 will be a zero of the left side of (2.13) with multiplicity $2p-1$ whereas that of the right side of (2.13) is $2p+(q-1)$. In order to have the same zeros with same multiplicities of both the sides of (2.13), we must have $2p-1 = 2p+(q-1)$, and we see that $q = 0$, which is a contradiction.

Since z_0 could not be a zero of $\alpha(z)$, hence we see from (2.13), z_0 will be a zero of left side of (2.13) with multiplicity $2p-1$ but that of the right side of (2.13) is $2p$, which is a contradiction.

Case c. Let $g(z)$ is non-periodic but $g(z)$ and $g(z+c)$ have common zeros with different multiplicities. Let z_0 would be a zero of $g(z)$ with multiplicity $p(\geqslant 1)$ and a zero of $g(z+c)$ with multiplicity $q(\geqslant 1)$ and $p \neq q$.

Now z_0 would be a zero of $\mathcal{L}_c(g)$ and g with respective multiplicities $\min\{p,q\}$ and p, and hence we see from (2.14) $\min\{p,q\} = p$, as $\mathcal{L}_c(g)$ and g share 0 CM. Thus we affirm that $p < q$. Again, by the same argument, it follows from (2.13) that $\min\{p,q\} = q-1$. Therefore, we must have $q = p+1$. Thus we see that z_0 be a zero of $g(z)$ and $g(z+c)$ with multiplicities p and $p+1$ respectively, and hence one can expresses $g(z)$ and $g(z+c)$ as follows

$$g(z) = (z-z_0)^p\phi(z) \ \text{ and } \ g(z+c) = (z-z_0)^{p+1}\psi(z) \tag{2.16}$$

where $\phi(z)$ and $\psi(z)$ are non-constant meromorphic functions ($\phi(z)$ may have pole at z_0 only with multiplicity at most $p-1$) such that $\phi(z_0) \neq 0, \ \infty$ and $\psi(z_0) \neq 0$. We note that $\phi(z)$ must contain the factor $(z-c-z_0)$, otherwise z_0 could not be a zero of $g(z+c)$.

From (2.14), we now obtained

$$e^{\alpha(z)} = c_1\frac{g(z+c)}{g(z)} + c_0 = c_1\frac{(z-z_0)\psi(z)}{\phi(z)} + c_0. \tag{2.17}$$

Equation (2.17) shows that z_0+c is a pole of $e^{\alpha(z)}$, which is clearly a contradiction.

Therefore, we have $c_1+c_0 \neq 1$, and based on this assumption, we consider the following two cases in our next discussions.

Case 1. If $\mathcal{G}(z) \equiv 0$, then the Eq. (2.12) becomes

$$\alpha'(z) + \frac{g'(z)}{g(z)} \equiv 0, \tag{2.18}$$

which in turn gives the solution

$$g(z) = \mathcal{C}e^{-\alpha(z)}, \tag{2.19}$$

where $\mathcal{C}$ is an arbitrary constants. We substitute (2.19) in (2.9) and obtain

$$\frac{(\mathcal{C}+1)-(c_0+c_1)}{\mathcal{C}}e^{\alpha(z)} = c_1e^{\alpha(z)-\alpha(z+c)} + c_0. \tag{2.20}$$

From the R.H.S of (2.20), it follows that α has to be a constant. Then we must have $c_1+c_0 = 0$, and so $\mathcal{C} = -1$, and then $e^{\alpha(z)} = e^{\alpha(z+c)}$. Therefore by (2.9), one can verify that $\mathcal{L}_c(f) \equiv 0$, a contradiction.

Case 2. Let $\mathcal{G}(z) \not\equiv 0$.

By Lemma 2.1, we get that $m(r, \mathcal{G}) = S(r, f)$. From (2.9), we see that if $g(z_0) = 0$ for some $z_0 \in \mathbb{C}$, then

$$g(z_0 + c) = \frac{1 - (c_0 + c_1)}{c_1},$$

so it follows that

$$N\left(r, \frac{g'(z)g(z+c)}{g^2(z)}\right) = \overline{N}\left(r, \frac{1}{g(z)}\right) + N\left(r, \frac{1}{g(z)}\right).$$

Clearly $N(r, \mathcal{G}) = \overline{N}\left(r, \frac{1}{g(z)}\right) + N\left(r, \frac{1}{g(z)}\right)$.

Therefore, since g is an entire function, we have

$$\begin{aligned} m(r,g) &\leqslant m\left(r, \frac{1}{\mathcal{G}}\right) + S(r,f) \\ &\leqslant N(r,\mathcal{G}) + S(r,f) \\ &= \overline{N}\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g}\right) + S(r,f) \\ &\leqslant T(r,g) - m\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g}\right) + S(r,f) \\ &\leqslant m(r,g) + N(r,g) - m\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g}\right) + S(r,f) \\ &\leqslant m(r,g) - m\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g}\right) + S(r,f), \end{aligned}$$

which in turn implies that

$$m\left(r, \frac{1}{g}\right) \leqslant \overline{N}\left(r, \frac{1}{g}\right) + S(r,f). \tag{2.21}$$

Again, we can re-write (2.9) as

$$c_1 \frac{g(z+c)}{g(z)} + \frac{c_0 + c_1 - 1}{g(z)} = e^{\alpha(z)} - c_0.$$

Hence, it is clear to us

$$m\left(r, e^{\alpha(z)}\right) = m\left(r, \frac{1}{g(z)}\right) + S(r,f), \tag{2.22}$$

possibly outside of finite logarithmic measure.

Applying *Second Fundamental Theorem* to the function, we obtained that

$$\overline{N}\left(r, \frac{1}{e^{\beta(z)} - 1}\right) = m\left(r, e^{\beta(z)}\right) + \left(r, e^{\beta(z)}\right).$$

Again note that from (2.8), we get

$$\overline{N}\left(r, \frac{1}{f-a}\right) \leqslant \overline{N}\left(r, \frac{1}{e^{\beta(z)}-1}\right) \tag{2.23}$$

Then from (2.21), (2.22) and (2.23), it follows that

$$\begin{aligned} m\left(r, e^{\alpha(z)}\right) &\leqslant \overline{N}\left(r, \frac{1}{f-a}\right) + S(r,f) \\ &\leqslant \overline{N}\left(r, \frac{1}{e^{\beta(z)}-1}\right) + S(r,f) \\ &\leqslant m\left(r, e^{\beta(z)}\right) + S(r,f). \end{aligned} \tag{2.24}$$

Similarly, we can obtained

$$\begin{aligned} m\left(r, e^{\beta(z)}\right) &\leqslant \overline{N}\left(r, \frac{1}{f-b}\right) + S(r,f) \\ &\leqslant \overline{N}\left(r, \frac{1}{e^{\alpha(z)}-1}\right) + S(r,f) \\ &\leqslant m\left(r, e^{\alpha(z)}\right) + S(r,f). \end{aligned} \tag{2.25}$$

By combining the inequalities (2.24) and (2.25), we have

$$m\left(r, e^{\beta(z)}\right) = m\left(r, e^{\alpha(z)}\right) + S(r,f). \tag{2.26}$$

Again solving $\mathcal{L}_c(f)$ and f, we get from (2.8) that

$$f(z) = \frac{b-a+ae^{\alpha(z)}-be^{\beta(z)}}{e^{\alpha(z)}-e^{\beta(z)}}. \tag{2.27}$$

Therefore from (2.8), (2.26) and (2.27), we get that

$$S(r,f) = S\left(r, e^{\alpha(z)}\right) = S\left(r, e^{\beta(z)}\right).$$

With the help of the standard relation in Nevanlinna Theory

$$m\left(r, e^{a_n z^n}\right) = \frac{|a_n| r^n}{\pi},$$

we obtain from (2.26) that

$$\frac{|\alpha_n| r^n}{\pi}(1+o(1)) = \frac{|\beta_m| r^m}{\pi}(1+o(1)),$$

which clearly implies that, $n=m$ and $|\alpha_n| = |\beta_m|$.

This completes the proof. □

Lemma 2.7. *Let f be an entire function and $\mathcal{L}_c(f) = c_1 f(z+c) + c_0 f(z)$ be such that*

$$\frac{\mathcal{L}_c(f)-a}{f-a} \equiv e^{\alpha} \quad \textit{and} \quad \frac{\mathcal{L}_c(f)-b}{f-b} \equiv e^{\beta},$$

where $ab \neq 0$ and rests are as in Lemma 2.6, then α, β, 2α, 2β, $\alpha + 2\beta$ and $\beta + 2\alpha$ all have the same degree n.

Proof. Since $ab \neq 0$, so proceeding exactly the same way as done in Lemma 2.6, we get $n = m$ with $|\alpha_n| = |\beta_m|$. It is now enough to show that the polynomial $\alpha + \beta$ has degree n. From the assumption, solving for f and $\mathcal{L}_c(f)$, we get

$$f(z) = \frac{b - a + ae^{\alpha(z)} - be^{\beta(z)}}{e^{\alpha(z)} - e^{\beta(z)}} \tag{2.28}$$

and

$$\mathcal{L}_c(f) = \frac{be^{\alpha(z)} - ae^{\beta(z)} + (a-b)e^{\alpha(z)+\beta(z)}}{e^{\alpha(z)} - e^{\beta(z)}}. \tag{2.29}$$

We also see that

$$\begin{aligned} \mathcal{L}_c(f) &= c_1 f(z+c) + c_0 f(z) \\ &= c_1 \frac{(b-a) + ae^{\Delta\alpha}e^{\alpha} - be^{\Delta\beta}e^{\beta}}{e^{\Delta\alpha}e^{\alpha} - e^{\Delta\beta}e^{\beta}} + c_0 \frac{b - a + ae^{\alpha(z)} - be^{\beta(z)}}{e^{\alpha(z)} - e^{\beta(z)}}. \end{aligned} \tag{2.30}$$

From (2.29) and (2.30), we get

$$\begin{aligned} &\frac{be^{\alpha} - ae^{\beta} + (a-b)e^{\alpha+\beta} - c_0(b-a) - c_0 ae^{\alpha} + c_0 be^{\beta}}{e^{\alpha} - e^{\beta}} \\ &\quad \equiv \frac{c_1(b-a) + c_a e^{\Delta\alpha}e^{\alpha} - c_1 be^{\Delta\beta}e^{\beta}}{e^{\Delta\alpha}e^{\alpha} - e^{\Delta\beta}e^{\beta}}, \end{aligned}$$

which in turn implies that

$$\begin{aligned} &\Big\{(b - c_0 a - c_1 a)e^{\Delta\alpha}\Big\}e^{2\alpha} - \Big\{(bc_0 - a + c_1 b)e^{\Delta\beta}\Big\}e^{2\beta} + \Big\{(a-b)e^{\Delta\alpha}\Big\}e^{2\alpha+\beta} \\ &\quad - \Big\{(a-b)e^{\Delta\beta}\Big\}e^{\alpha+2\beta} + \Big\{(bc_0 + c_a - a)e^{\Delta\alpha} + (c_1 b + c_0 a - b)e^{\Delta\beta}\Big\}e^{\alpha+\beta} \\ &\quad - (b-a)\Big\{c_0 e^{\Delta\alpha} + c_1\Big\}e^{\alpha} + (b-a)\Big\{c_0 e^{\Delta\beta} + c_1\Big\}e^{\beta} \equiv 0. \end{aligned} \tag{2.31}$$

Next if the degree of the polynomial $\alpha+\beta$ is less than n, then it immediate that, in the two polynomials $\alpha(z)$ and $\beta(z)$, the relation between the leading coefficients α_n and β_n (since $n = m$) respectively, must be $\alpha_n + \beta_n = 0$. Therefore, we can write

$$\begin{aligned} \alpha(z) &= \alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0, \\ \beta(z) &= -\alpha_n z^n + \beta_{m-1} z^{m-a} + \cdots + \beta_1 z + \beta_0. \end{aligned}$$

Substituting this in (2.31), we get that

$$\begin{aligned}
&\Big\{(b-c_0a-c_1a)e^{\Delta\alpha}\Big\}e^{p_1(z)}e^{2\alpha_n z^n}-\Big\{(bc_0-a+c_1b)e^{\Delta\beta}\Big\}e^{p_2(z)}e^{-2\alpha_n z^n}\\
&\quad+\Big\{(a-b)e^{\Delta\alpha}\Big\}e^{p_3(z)}e^{\alpha_n z^n}-\Big\{(a-b)e^{\Delta\beta}\Big\}e^{p_4(z)}e^{-\alpha_n z^n}\\
&\quad+\Big\{(bc_0+c_a-a)e^{\Delta\alpha}+(c_1b+c_0a-b)e^{\Delta\beta}\Big\}e^{p_5(z)}\\
&\quad-(b-a)\Big\{c_0e^{\Delta\alpha}+c_1\Big\}e^{p_6(z)}e^{\alpha_n z^n}+(b-a)\Big\{c_0e^{\Delta\beta}+c_1\Big\}e^{p_7(z)}e^{-\alpha_n z^n}\equiv 0,
\end{aligned}\tag{2.32}$$

where $P_j(z)$ $(j=1,2,\ldots,7)$ all are polynomials with $\deg(P_j(z))<n$.

We now define a function $\mathcal{H}(z):=e^{\alpha_n z^n}$, and with the help of this, we can rewrite (2.32) as follows

$$\begin{aligned}
&\Big\{(b-c_0a-c_1a)e^{\Delta\alpha}\Big\}\mathcal{H}^4+\Big\{(a-b)e^{\Delta\alpha}e^{P-3(z)}\}\mathcal{H}^3\\
&\quad+\Big\{\Big[(bc_0+c_0a-a)e^{\Delta\alpha}+(c_1b+c_0a-b)e^{\Delta\beta}\Big]e^{P_5(z)}\Big\}\mathcal{H}^2\\
&\quad+(b-a)\Big\{e^{\Delta\beta}e^{P_4(z)}-\Big[c_0e^{\Delta\alpha}+c_1\Big]e^{P_6(z)}+\Big[c_0e^{\Delta\beta}+c_1\Big]e^{P_7(z)}\Big\}\mathcal{H}\\
&\quad+\Big[-(bc_0+c_1b-a)e^{\Delta\beta}\Big]e^{P_2(z)}\equiv 0.
\end{aligned}\tag{2.33}$$

Applying Lemma 2.2 to the Eq. (2.33), we get

$$4\,T(r,\mathcal{H})=S(r,f),$$

which leads to a contradiction.

Thus we must have $\deg(\alpha+\beta)=n$. $\square$

Lemma 2.8. [17] *Let h_1, h_2, ..., h_p be linearly independent meromorphic functions satisfying*

$$h_1+h_2+\cdots+h_p=1.$$

Then, for $j=1,2,\ldots,p$, we have

$$T(r,h_j)\leqslant\sum_{k=1}^{p}N\left(r,\frac{1}{h_k}\right)-\sum_{k=1,k\neq j}^{p}N(r,h_k)+N(r,\mathcal{W})-N\left(r,\frac{1}{\mathcal{W}}\right)+S(r),$$

where $\mathcal{W}=\mathcal{W}(h_1,h_2,\ldots,h_p)$ is the Wronskian of h_1, h_2, ..., h_p, and

$$S(r)=O(\log r)+O(\log\max_{1\leqslant k\leqslant p}T(r,h_j)),\quad \textit{for } r\to\infty,\ r\in E,$$

for a set $E\subset(0,\infty)$ of finite Lebesgue measure. If all h_k have finite order, E can be chosen to be the empty set.

3. Proofs of the Main Results

Proof of Theorem 1.1. Since f is an entire function, and with $\mathcal{L}_c(f)$ share a, b **CM**, also f being a finite ordered, so there must exit two polynomials $\alpha \equiv \alpha(z)$ and $\beta \equiv \beta(z)$, such that

$$\frac{\mathcal{L}_c(f) - a}{f - a} = e^{\alpha} \quad \text{and} \quad \frac{\mathcal{L}_c(f) - b}{f - b} = e^{\beta}. \tag{3.1}$$

Case 1. If $e^{\alpha} = e^{\beta}$, then from (3.1), we deduce that

$$\frac{\mathcal{L}_c(f) - a}{f - a} = \frac{\mathcal{L}_c(f) - b}{f - b},$$

which in turn implies that $\mathcal{L}_c(f) \equiv f$.

We are now at a position to find the class of all the entire functions satisfying the relation $\mathcal{L}_c(f) \equiv f$. By the assumption of the result, and using Lemma 2.5, we see that f is not a rational function. Therefore $f(z)$ must be a transcendental entire function.

Next we rewrite the above relation as follows

$$f(z + c) = \left(\frac{1 - c_0}{c_1}\right) f(z). \tag{3.2}$$

Let $f_1(z)$ and $f_2(z)$ are the two solutions of (3.2). Then we have

$$f_1(z + c) = \left(\frac{1 - c_0}{c_1}\right) f_1(z) \tag{3.3}$$

$$f_2(z + c) = \left(\frac{1 - c_0}{c_1}\right) f_2(z). \tag{3.4}$$

Let us set a function $\phi(z) := f_1(z)/f_2(z)$. Then with the help of (3.3) and (3.4), we get

$$\phi(z + c) = \frac{f_1(z + c)}{f_2(z + c)} = \frac{\dfrac{1 - c_0}{c_1} f_1(z)}{\dfrac{1 - c_0}{c_1} f_2(z)} = \frac{f_1(z)}{f_2(z)} = \phi(z),$$

for all $z \in \mathbb{C}$. It can be easily check that the function $f_2(z) = \left(\frac{1-c_0}{c_1}\right)^{\frac{z}{c}} g_2(z)$, where $g_2(z)$ is a entire function with $g_2(z + c) = g_2(z)$, is a solution of (3.5). Also, it is easy to verify that $f_1(z) = f_2(z)\phi(z)$, a solution of (3.5).

Thus, one can observe that the linear combination

$$\begin{aligned}\mathcal{L}_c(f) &= a_1 f_1(z) + a_2 f_2(z)\\ &= \left(\frac{1-c_0}{c_1}\right)^{\frac{z}{c}} (a_1\phi(z) + a_2)\, g_2(z)\\ &= \left(\frac{1-c_0}{c_1}\right)^{\frac{z}{c}} \sigma(z),\end{aligned}$$

where $\sigma(z) = (a_1\phi(z) + a_2)\, g_2(z)$ is such that $\sigma(z+c) = \sigma(z)$, for all $z \in \mathbb{C}$, is the general solution of (3.5). Hence, we conclude that $f(z)$ must assume the following form

$$f(z) = \left(\frac{1-c_0}{c_1}\right)^{\frac{z}{c}} g(z),$$

where $g(z)$ is a entire function with $g(z+c) = g(z)$, for all $z \in \mathbb{C}$.

Case 2. If both α and β are constants and $e^{\alpha} \neq e^{\beta}$, then from (3.1) we see that f is a constant, which is not possible.

So in this case, we have $e^{\alpha} = e^{\beta}$, and then it follows from (3.1) that $\mathcal{L}_c(f) \equiv f$. Proceeding exactly same way as done in case 1, we get $f(z)$ must assume the following form

$$f(z) = \left(\frac{1-c_0}{c_1}\right)^{\frac{z}{c}} g(z),$$

where $g(z)$ is a entire function with $g(z+c) = g(z)$, for all $z \in \mathbb{C}$.

Therefore, we just to consider the case that at least one of α and β is non-constant, and $e^{\alpha} \neq e^{\beta}$. Next solving for $\mathcal{L}_c(f)$ and f, from (3.1), we obtained that

$$f(z) = \frac{b - a + ae^{\alpha(z)} - be^{\beta(z)}}{e^{\alpha(z)} - e^{\beta(z)}} \tag{3.5}$$

and

$$\mathcal{L}_c(f) = \frac{be^{\alpha(z)} - ae^{\beta(z)} + (a-b)e^{\alpha(z)+\beta(z)}}{e^{\alpha(z)} - e^{\beta(z)}}. \tag{3.6}$$

If only one of $\alpha(z)$ and $\beta(z)$ is constant, then without any loss of generality, we assume that $\beta(z)$ is a constant. Then from (3.1), we obtained that if z_0 is a zero of $e^{\alpha(z)-\beta(z)}$, then we have $e^{\alpha(z_0)} = e^{\beta(z_0)} = 1$. Then $e^{\beta(z)} \neq 1$ implies that $e^{\alpha(z)} \neq e^{\beta(z)}$, we know that this is impossible from the *Second Fundamental Theorem*.

If $e^{\beta} = 1$, then we obtained that $\mathcal{L}_c(f) \equiv f$ from (3.1), and hence as above, we get

$$f(z) = \left(\frac{1-c_0}{c_1}\right)^{\dfrac{z}{c}} g(z),$$

where $g(z)$ is a entire function with $g(z+c) = g(z)$, for all $z \in \mathbb{C}$.

Case 3. Suppose that neither $\alpha(z)(\not\equiv \beta(z))$ nor $\beta(z)$ is a constant. Then from (3.5), we get

$$\overline{N}\left(r, \frac{1}{f-a}\right) \leqslant \overline{N}\left(r, \frac{1}{e^{\beta(z)}-1}\right) \tag{3.7}$$

and

$$\overline{N}\left(r, \frac{1}{f-b}\right) \leqslant \overline{N}\left(r, \frac{1}{e^{\alpha(z)}-1}\right). \tag{3.8}$$

Now we are at a position to discuss the next two subcases.

Subcase 3.1. Let $ab \neq 0$. We set here

$$\alpha(z) = \alpha_n z^n + \cdots + \alpha_1 z + \alpha_0$$

and

$$\beta(z) = \beta_m z^n + \cdots + \beta_1 z + \beta_0,$$

where $\alpha_n(\neq 0), \alpha_{n-1}, \ldots, \alpha_1, \alpha_0 \in \mathbb{C}$ and $\beta_m(\neq 0), \beta_{m-1}, \ldots, \beta_1, \beta_0 \in \mathbb{C}$.

Next we have the following subcases to discuss.

Subcase 3.1.1. We claim that $m = n$ and $|\alpha_n| = |\beta_m|$. This follows directly from Lemma 2.6.

Subcase 3.1.2. In this case, we need to show that the polynomials α, β, 2α 2β, $\alpha+\beta$, $2\alpha+\beta$, $\alpha+2\beta$ are the polynomials of same degree n. With the help of Lemma 2.7,one can get it easily.

Subcase 3.1.3. In this case we want to show that α, β, $\alpha-\beta$, $2\alpha-\beta$, $\alpha-2\beta$, $\ldots$, $\beta-2\alpha$, which means $g_j - g_k$ in Lemma 2.4, are all polynomials with n.

Next, from the *Subcases 3.1.1* and *3.1.2*, we just need to consider the case of the polynomial $\alpha-\beta$. Let if possible $\deg(\alpha-\beta) < n$. Therefore, we observe that the relation between the leading coefficients α_n in $\alpha(z)$ and β_n in $\beta(z)$ is $\alpha_n - \beta_n = 0$. Therefore $\alpha(z)$ and $\beta(z)$ can be written as

$$\alpha(z) = \alpha_n z^n + \cdots + \alpha_1 z + \alpha_0.$$
$$\beta(z) = \alpha_n z^n + \cdots + \beta_1 z + \beta_0.$$

Next we set $P(z) = \alpha(z) - \beta(z)$, and hence one can check that $\deg(P) < n$. As in Lemma 2.7, here also we can get

$$\begin{aligned} &\left\{(b - c_0a - c_1a)e^{\Delta\alpha}\right\}e^{2\alpha} - \left\{(bc_0 - a + c_1b)e^{\Delta\beta}\right\}e^{2\beta} + \left\{(a-b)e^{\Delta\alpha}\right\}e^{2\alpha+\beta} \\ &\quad - \left\{(a-b)e^{\Delta\beta}\right\}e^{\alpha+2\beta} + \left\{(bc_0 + c_a - a)e^{\Delta\alpha} + (c_1b + c_0a - b)e^{\Delta\beta}\right\}e^{\alpha+\beta} \\ &\quad - (b-a)\left\{c_0e^{\Delta\alpha} + c_1\right\}e^{\alpha} + (b-a)\left\{c_0e^{\Delta\beta} + c_1\right\}e^{\beta} \equiv 0. \end{aligned} \tag{3.9}$$

Substituting $\alpha(z) = P(z) + \beta(z)$ in (3.9), we get

$$\begin{aligned} &(a-b)\left\{e^{\Delta\alpha}e^{2p} - e^{\Delta\beta}e^{P}\right\}e^{3\beta} \\ &\quad + \left\{(b - c_0a - c_1a)e^{\Delta\alpha}e^{2P} + \left[(bc_0 + c_1a - a)e^{\Delta\alpha} + (c_1b + c_0a - b)e^{\Delta\beta}\right]e^{P}\right\}e^{2\beta} \\ &\quad + (b-a)\left\{-(c_0e^{\Delta\alpha} + c_1)e^{P} + (c_0e^{\Delta\beta} + c_1)\right\}e^{\beta} \equiv 0. \end{aligned} \tag{3.10}$$

This can be written as

$$\sum_{j=1}^{3} f_j(z)e^{g_j(z)} \equiv 0, \tag{3.11}$$

where $g_j(z) = j\beta(z)$, and

$$\begin{aligned} f_1(z) &= (b-a)\left\{-(c_0e^{\Delta\alpha} + c_1)e^{P} + (c_0e^{\Delta\beta} + c_1)\right\}. \\ f_2(z) &= \left\{(b - c_0a - c_1a)e^{\Delta\alpha}e^{2P} + \left[(bc_0 + c_1a - a)e^{\Delta\alpha} + (c_1b + c_0a - b)e^{\Delta\beta}\right]e^{P}\right\}. \\ f_3(z) &= (a-b)\left\{e^{\Delta\alpha}e^{2p} - e^{\Delta\beta}e^{P}\right\}. \end{aligned}$$

Applying Lemma 2.4, we obtained that $f_1(z) \equiv 0$, $f_2(z) \equiv 0$ and $f_3(z) \equiv 0$. Now $f_1(z) \equiv 0$ implies that

$$e^{P} = \frac{c_1 + c_0e^{\Delta\beta}}{c_1 + e^{\Delta\alpha}}. \tag{3.12}$$

And $f_3(z) \equiv 0$ implies that

$$e^{P} = \frac{e^{\Delta\beta}}{e^{\Delta\alpha}}. \tag{3.13}$$

Therefore combining (3.12) and (3.13), we get $e^{\Delta\alpha} = e^{\Delta\beta}$ which in turn implies that $e^{\alpha} = e^{\beta}$, and this is not possible as it is the omitted case. Thus it follows from *Subcases 3.1.1* and *3.1.2*, and from (3.9) and Lemma 2.4 that

$$c_1+c_0e^{\Delta\alpha}=0,\ c_1+c_0e^{\Delta\beta}=0,\ (b-c_0a-c_1a)e^{\Delta\alpha}=0,$$
$$(bc_0+bc_1-a)e^{\Delta\beta}=0,(bc_0+c_1a-a)e^{\Delta\alpha}+(c_1b+c_0a-b)e^{\delta\beta}=0,$$
$$e^{\Delta\alpha}=0,\ e^{\delta\beta}=0,$$

which is not possible.

Case 3.2. Let $ab=0$. This implies that either $a=0$ or $b=0$ as $a\neq b$. Without any loss of generality, we may suppose that $a=1$ and $b=0$. Then Eq. (3.1) reduces to

$$c_1\frac{f(z+c)}{f(z)}=e^{\beta}-c_0. \tag{3.14}$$

and

$$f(z)=\frac{e^{\alpha(z)}-1}{e^{\alpha(z)}-e^{\beta}(z)}. \tag{3.15}$$

Applying Lemma 2.1, we see from (3.14), we get

$$m\left(r,e^{\beta}\right)=o\left\{r^{\rho(z)-1+\epsilon}\right\}.$$

i.e., we see that

$$\rho\left(e^{\beta}\right)\leqslant\rho(f)-1<\rho(f).$$

Thus, from (3.15), and the above equation, we obtained that $T(r,f)=T\left(r,e^{\alpha}\right)+S(r,f)$. i.e., $T\left(r,e^{\beta}\right)=S\left(r,e^{\alpha}\right)$. Again from (3.15), we see that $e^{\alpha}-e^{\beta}=0$. i.e., $e^{\alpha}=e^{\beta}=1$, which shows that

$$\begin{aligned}\overline{N}\left(r,\frac{1}{e^{\alpha}-e^{\beta}}\right)&\leqslant N\left(r,\frac{1}{e^{\beta}-1}\right)\\&=S\left(r,e^{\beta}\right).\end{aligned}$$

Therefore, by the *Second Fundamental Theorem* for small function, we get

$$T\left(r,e^{\alpha}\right)\leqslant\epsilon T\left(r,e^{\alpha}\right)+S\left(r,e^{\alpha}\right).$$

i.e.,

$$(1-\epsilon)\,T\left(r,e^{\alpha}\right)\leqslant S\left(r,e^{\alpha}\right),$$

which is clearly absurd as ϵ is an arbitrary positive number.

This completes the proof of *Theorem 1.1.* □

Proof of Theorem 1.2. Since f and $\mathcal{L}_c(f)$ share a, b, ∞ **CM**, and f is of finite ordered meromorphic function. Then there exists two polynomials $\alpha\equiv\alpha(z)$, $\beta\equiv\beta(z)$ such that

$$\frac{f-a}{\mathcal{L}_c(f)-a}=e^{\alpha},\quad\frac{f-b}{\mathcal{L}_c(f)-b}=e^{\beta}. \tag{3.16}$$

We now discuss the following possible cases.

Case 1. If $e^{\alpha} = 1$ or $e^{\beta} = 1$, then from (3.16), we get that $\mathcal{L}_c(f) \equiv f(z)$. So proceeding exactly same way as done in the proof of Theorem 1.1, we can show that the meromorphic function f here takes the form

$$f(z) = \left(\frac{1-c_0}{c_1}\right)^{\frac{z}{c}} g(z),$$

where $g(z)$ is a meromorphic function with $g(z+c) = g(z)$, for all $z \in \mathbb{C}$.

Case 2. Let neither $e^{\alpha} = 1$, nor $e^{\beta} = 1$ but $e^{\alpha} = e^{\beta}$. Then we get from (3.16),

$$\frac{f-a}{\mathcal{L}_c(f)-a} = \frac{f-b}{\mathcal{L}_c(f)-b},$$

which shows that $\mathcal{L}_c(f) \equiv f$, and as above we get the form of the function is

$$f(z) = \left(\frac{1-c_0}{c_1}\right)^{\frac{z}{c}} g(z),$$

where $g(z)$ is a meromorphic function with $g(z+c) = g(z)$, for all $z \in \mathbb{C}$.

Case 3. We suppose that neither $e^{\alpha} = 1$, nor $e^{\beta} = 1$ nor $e^{\alpha} = e^{\beta}$.

Then solving for f and $\mathcal{L}_c(f)$, we get from (3.16) that

$$f(z) = a + (b-a)\frac{e^{\beta}-1}{e^{\gamma}-1} \tag{3.17}$$

and

$$\mathcal{L}_c(f) = b + (b-a)\frac{1-e^{-\alpha}}{e^{\gamma}-1}, \tag{3.18}$$

where $\gamma = \beta - \alpha$.

It follows from (3.17) that

$$T(r,f) \leqslant T\left(r, e^{\beta} + e^{\gamma}\right) + S(r,f).$$

Again we see that

$$\begin{aligned} \mathcal{L}_c(f) &= c_1 f(z+c) + c_0 f(z) \\ &= a(c_1+c_0) + (b-a)\left\{c_1\frac{e^{\beta(z+c)}-1}{e^{\gamma(z+c)}-1} + c_0\frac{e^{\beta}-1}{e^{\gamma}-1}\right\} \\ &= a(c_1+c_0) + (b-a)\left\{c_1\frac{\beta_1 e^{\beta}-1}{\gamma_1 e^{\gamma}-1} + c_0\frac{e^{\beta}-1}{e^{\gamma}-1}\right\}, \end{aligned} \tag{3.19}$$

where $\beta_1(z) = e^{\beta(z+c)-\beta(z)}$, $\gamma_1(z) = e^{\gamma(z+c)-\gamma(z)}$. Our main aim is to prove that $\deg(\beta) = \deg(\gamma)$ and for this we our discussion will includes the following two cases.

Subcase 3.1. Let us suppose that $\deg(\beta) < \deg(\gamma)$. Therefore, we see that e^{β} is a small function of e^{γ}. Again note that $\deg[\beta(z+c)-\beta(z)] \leqslant \deg(\beta) < \deg(\gamma)$ and $\deg[\gamma(z+c)-\gamma(z)] < \deg(\gamma)$, which implies that $\beta_1(z) = e^{\beta(z+c)-\beta(z)}$ and $\gamma_1 = e^{\gamma(z+c)-\gamma(z)}$ are small functions of e^{γ}.

Let z_0 be a zero of $\gamma_1 e^{\gamma} - 1$, and $\beta_1(z_0)e^{\beta(z_0} - 1 \neq 0$. If $e^{\gamma(z_0)} - 1 \neq 0$, then clearly z_0 would be a pole of $\mathcal{L}_c(f)$, which in turn shows from (3.18) that z_0 would be a zero of $e^{\gamma} - 1$. Thus, we conclude that z_0 is a zero of $e^{\gamma} - 1$.

If $\gamma_1 - 1 \neq 0$, then applying *Second Fundamental Theorem*, we obtained that

$$\begin{aligned} T\left(r, e^{\gamma}\right) &\leqslant \overline{N}\left(r, \frac{1}{\gamma_1 e^{\gamma}-1}\right) + \overline{N}\left(r, \frac{1}{e^{\gamma}}\right) + \overline{N}\left(r, e^{\gamma}\right) + S(r, e^{\gamma}) \\ &\leqslant N\left(r, \frac{1}{\beta_1 e^{\beta}-1}\right) + N\left(r, \frac{1}{\gamma_1 - 1}\right) + S(r, e^{\gamma}) \\ &\leqslant S(r, e^{\gamma}), \end{aligned}$$

which is impossible.

Thus we must have $\gamma_1 - 1 = 0$. i.e., $e^{\gamma(z+c)-\gamma(z)} = 1$, and which shows that $\deg(\gamma) = 1$. Since $\deg(\beta) < \deg(\gamma)$, so one can immediately get that β would be a constant, say $\mathcal{B}$. So, we get from (3.19) that

$$\mathcal{L}_c(f) = a(c_1 + c_0) + (b-a)(c_1+c_0)\left(\frac{e^{\beta}-1}{e^{\gamma}-1}\right),$$

which contradicts that $\mathcal{L}_c(f)$ is not of the form $\dfrac{d_1 e^{\alpha} + d_2}{d_3 e^{\beta} + d_4}$, where d_1, d_2, $d_3, d_4 \in \mathbb{C}$, and α, β are two polynomials in z.

Subcase 3.2. Let $\deg(\beta) > \deg(\gamma)$.

In this case, e^{γ} is a small function of e^{β}. Now proceeding as in *Subcase 3.1*, we can get that β_1, γ_1 are small function of e^{β}.

Let z_1 be a zero of $e^{\beta} - 1$, such that $e^{\gamma(z_1)} - 1 \neq 0$. Then z_1 be a zero of $f - a$. Since f and $\mathcal{L}_c(f)$ share a **CM**, so it is clear that z_1 is also a zero of $\mathcal{L}_c(f)$. Now from (3.19), we see that

$$\mathcal{L}_c(f) - a = a(c_1+c_0) + (b-a)\left\{c_1\frac{\beta_1 e^{\beta}-1}{\gamma_1 e^{\gamma}-1} + c_0\frac{e^{\beta}-1}{e^{\gamma}-1}\right\} - a.$$

Therefore, we get

$$(b-a)c_1\frac{\beta_1(z_1)-1}{\gamma_1(z_1)e^{\gamma(z_1)}-1} = a(1-c_1-c_0).$$

We now deduce that

$$(b-a)c_1\frac{\beta_1(z)-1}{\gamma_1(z)e^{\gamma(z)}-1} = a(1-c_1-c_0).$$

Otherwise, note that $e^{\gamma}-1\neq 0$ since $e^{\alpha}\neq e^{\beta}$. Applying *Second Fundamental Theorem*, we obtained that

$$\begin{aligned}T\left(r,e^{\beta}\right) &\leqslant \overline{N}\left(r,\frac{1}{e^{\beta}-1}\right)+\overline{N}\left(r,\frac{1}{e^{\beta}}\right)+\overline{N}\left(r,e^{\beta}\right)+S\left(r,e^{\beta}\right)\\ &\leqslant N\left(r,\frac{1}{e^{\beta}-1}\right)+S\left(r,e^{\beta}\right)\\ &\leqslant S\left(r,e^{\beta}\right),\end{aligned}$$

which is not possible. Thus we have

$$(b-a)c_1\frac{\beta_1-1}{\gamma_1 e^{\gamma}-1}=a(1-c_1-c_0).$$

i.e.,

$$c_1(b-a)e^{\beta(z+c)-\beta(z)}-c_1(b-a)=a(1-c_1-c_0)e^{\gamma(z)}-a(1-c_0-c_1).$$

Our aim is to show that γ is a constant. On contrary, we suppose that $\deg(\gamma)\geqslant 1$. Since $\deg(\beta)>\deg(\gamma)$, so we get

$$\begin{aligned}c_1(b-a)e^{\beta(z+c)-\beta(z)}&=a(1-c_0-c_1)e^{\gamma(z)}\\ \text{and}\quad c_1(b-a)&=a(1-c_1-c_0),\end{aligned}$$

which implies that $\beta_1=e^{\beta(z+c)-\beta(z)}=e^{\gamma(z)}$, for all $z\in\mathbb{C}$. Combining (3.18) and (3.19), we get

$$(b-a)\left\{c_1\frac{\beta_1 e^{\beta}-1}{\gamma_1 e^{\gamma}-1}+c_0\frac{e^{\beta}-1}{e^{\gamma}-1}\right\}=(b-a)\frac{1-e^{\gamma-\beta}}{e^{\gamma}-1}+b-a(c_1+c_0),$$

which in turn implies that

$$\begin{aligned}&\left\{c_1\beta_1\left(e^{\gamma}-1\right)+c_0\left(\gamma_1 e^{\gamma}-1\right)\right\}e^{2\beta}\\ &\quad+\left\{-c_1\left(e^{\gamma}-1\right)-\left(c_0+1\right)\left(\gamma_1 e^{\gamma}-1\right)-\frac{b-a(c_0+c_1)}{b-a}\left(e^{\gamma}-1\right)\left(\gamma_1 e^{\gamma}-1\right)\right\}e^{\beta}\\ &\quad+e^{\gamma}\left(\gamma_1 e^{\gamma}-1\right)\equiv 0.\end{aligned}\tag{3.20}$$

We see that (3.20) is the form

$$\sigma_2 e^{2\beta}+\sigma_1 e^{\beta}+\sigma_0\equiv 0,\tag{3.21}$$

where

$$\begin{aligned}\sigma_2(z)&=c_1\beta_1\left(e^{\gamma}-1\right)+c_0\left(\gamma_1 e^{\gamma}-1\right).\\ \sigma_1(z)&=-c_1\left(e^{\gamma}-1\right)-\left(c_0+1\right)\left(\gamma_1 e^{\gamma}-1\right)-\frac{b-a(c_0+c_1)}{b-a}\left(e^{\gamma}-1\right)\left(\gamma_1 e^{\gamma}-1\right).\\ \sigma_0(z)&=e^{\gamma}\left(\gamma_1 e^{\gamma}-1\right).\end{aligned}$$

One can note that $\sigma_0,\sigma_1,\sigma_2$ are all small functions of e^{β} as $\deg(\beta)>\deg(\gamma)$.

Now from (3.21), we see that $\sigma_2(z) \equiv 0$. i.e.,

$$c_1\beta_1\left(e^{\gamma}-1\right) \equiv -c_0\left(\gamma_1 e^{\gamma}-1\right). \tag{3.22}$$

We have $\beta_1 = e^{\gamma(z+c)}$, hence using this in (3.22), we get

$$c_1 e^{\gamma(z+c)+\gamma(z)} + (c_0-c_1)e^{\gamma(z+c)} - c_0 \equiv 0, \tag{3.23}$$

which implies that γ is a constant. In other words, $\deg(\gamma)=0$, contradicts the fact that $\deg(\gamma) \geqslant 1$. Then it follows from Theorem 1.1, $\mathcal{L}_c(f) \equiv f$, and which contradicts our assumption $\mathcal{L}_c(f) \not\equiv f$.

Thus from all the above discussions, we conclude that $\deg(\beta)=\deg(\gamma)$. Based on the fact that f is not a constant function, so we can now assume that $\deg(\beta)=\deg(\gamma)=n \geqslant 1$.

Next simplifying Eq. (3.20), we get

$$\begin{aligned}&\gamma_1 e^{2\gamma} + \left\{\frac{b-a(c_1+c_0)}{a-b}\gamma_1\right\}e^{2\gamma+\beta} + \left\{(\gamma_1+1)\frac{b-a(c_1+c_0)}{b-a} - (c_0+1)\gamma_1 - c_1\right\}e^{\gamma+\beta}\\ &\quad + (-c_1\beta_1 - c_0)e^{2\beta} + (c_1\beta_1 + c_0\gamma_1)e^{2\beta+\gamma} + \left\{c_1 + c_0 + 1 - \frac{b-a(c_0+c_1)}{b-a}\right\}e^{\beta}\\ &\quad + (-1)e^{\gamma} \equiv 0.\end{aligned}$$

We can write the above equation in the following form

$$b_0 e^{2\gamma} + b_1 e^{\beta+2\gamma} + b_2 e^{\beta+\gamma} + b_3 e^{2\beta} + b_4 e^{2\beta+\gamma} + b_5 e^{\beta} + b_6 e^{\gamma} \equiv 0, \tag{3.24}$$

where

$$\begin{aligned} b_0 &= \gamma_1;\\ b_1 &= \frac{b-a(c_1+c_0)}{a-b}\gamma_1;\\ b_2 &= (\gamma_1+1)\frac{b-a(c_1+c_0)}{b-a} - (c_1+1)\gamma_1 - c_1;\\ b_3 &= -c_1\beta_1 - c_0;\\ b_4 &= c_1\beta_1 + c_0\gamma_1;\\ b_5 &= c_1 + c_0 + 1 - \frac{b-a(c_1+c_0)}{b-a};\\ b_6 &= -1. \end{aligned}$$

One can easily verify that all the b_j $(j=0,1,\ldots,6)$ are small function of e^{β} as well as e^{γ}. Equation (3.24) can be written as

$$\sum_{j=0}^{6} b_j e^{g_j} \equiv 0, \tag{3.25}$$

where $g_0 = 2\gamma$, $g_1 = 2\gamma+\beta$, $g_2 = \gamma+\beta$, $g_3 = 2\beta$, $g_4 = 2\beta+\gamma$, $g_5 = \beta$, $g_6 = \gamma$.

Our aim is to show that $\deg(\gamma-\beta)=n$. On the contrary, we suppose that $\deg(\gamma-\beta)<n$. Then, it is obvious that $e^{\gamma-\beta}$ is a small function of e^{β} and e^{γ}.

We denote $N_E\left(r, 1, e^{\alpha}, e^{\beta}\right)$ as the counting function of all the common zeros of $e^{\beta}-1$ and $e^{\gamma}-1$.

Since $e^{\beta} \neq e^{\gamma}$, so we have $e^{\beta-\gamma} \neq 1$. Thus we have

$$N_E\left(r, 1; e^{\beta}, e^{\gamma}\right) \leqslant N\left(r, \frac{1}{e^{\gamma-\beta}-1}\right) = S\left(r, e^{\gamma}\right).$$

Again, note that e^{γ} is of finite order, and hence we must have $S\left(r+|c|, e^{\gamma}\right)$ $= S\left(r, e^{\gamma}\right)$. We choose $z_2 \in \mathbb{C}$ in such a way that $\gamma_1(z_2)e^{\gamma(z_2+c)}-1=0$ but $\beta_1(z_2)e^{\beta(z_2+c)}-1 \neq 0$. From our previous discussions, we can get that z_2 is a zero of $e^{\gamma}-1$. Furthermore, z_2 is also a zero of γ_1-1.

If $\gamma_1-1 \neq 0$, then applying *Second Fundamental Theorem*, we have

$$\begin{aligned} T\left(r, e^{\gamma}\right) &\leqslant \overline{N}\left(r, \frac{1}{\gamma_1 e^{\gamma}-1}\right) + \overline{N}\left(r, \frac{1}{e^{\gamma}-1}\right) + \overline{N}\left(r, e^{\gamma}\right) \\ &\leqslant N_E\left(r+|c|, 1, e^{\alpha}, e^{\beta}\right) + N\left(r, \frac{1}{\gamma_1-1}\right) + S\left(r, e^{\gamma}\right) \\ &\leqslant T\left(r, \frac{1}{\gamma_1-1}\right) + S\left(r+|c|, e^{\gamma}\right) + S\left(r, e^{\gamma}\right) \\ &\leqslant S\left(r, e^{\gamma}\right), \end{aligned}$$

which is clearly absurd.

Thus we have $\gamma_1 = 1$, which implies that $\deg(\gamma) = 1$. Therefore, as a consequence, we see that $\deg(\beta-\gamma) < 1$. i.e., in other words $\beta-\gamma = \text{constant}$, say $\mathcal{D}$.

Since $e^{\gamma(z+c)-\gamma(z)} = 1$, so we must have

$$\begin{aligned} e^{\beta(z+c)-\beta(z)} &= e^{\beta(z+c)-\gamma(z+c)-\beta(z)+\gamma(z)} \\ &= e^{[\beta(z+c)-\gamma(z+c)]--[\beta(z)-\gamma(z)]} \\ &= e^{\mathcal{D}-\mathcal{D}} \\ &= 1. \end{aligned}$$

Then it follows from (3.19) that

$$\mathcal{L}_c(f) = (c_0+c_1)\left\{a + (b-a)\frac{e^{\beta}-1}{e^{\gamma}-1}\right\},$$

which contradicts our assumption $\mathcal{L}_c(f) \neq \frac{d_1 e^{\alpha}+d_2}{d_3 e^{\beta}+d_4}$, for some $d_j \in \mathbb{C}$, and for polynomials α and β in z.

Thus we have $\deg(\gamma-\beta) = n$. Further, one has $\deg(g_2-g_j) = n$ for $= 0, 1, 2, 4, 5, 6$. Now we assume that $b_3 = -c_1\beta_1 - c_0 \neq 0$. Thus we assume $\phi_j = b_j e^{g_j}$ $(j = 0, 1, \ldots, 6)$. Thus from (3.24), we have $\sum_{j=0}^{6} \phi_j \equiv 0$. Thus from the basic Linear Algebra, we deduce that there exist a set $\mathcal{J} \subset \{0, 1, 2, \ldots, 6\}$ and constants $\lambda_j(\neq 0) \in \mathbb{C},\ j \in \mathcal{J}$ such that $\phi_3 = \sum_{j\in\mathcal{J}} \lambda_j \phi_j$, and such that the set $\{\phi_j : j \in \mathcal{J}\}$ is linearly independent.

So, after re-writing, we must have:

$$\sum_{j\in\mathcal{J}}\lambda_j\frac{b_j}{b_3}e^{g_j-g_2}=1.$$

Applying Nevanlinna lemma to the function

$$h_j:=\lambda_j\frac{b_j}{b_3}e^{g_j-g_3},\quad j\in\mathcal{J},$$

which are linearly independent satisfying $\sum_{j\in\mathcal{J}}h_j=1$. We use the fact that zeros and poles of h_j, and their Wronskian can come only that of the function b_j with the following property

$$T(r,b_j)=O\left(r^{n-1}\right)=S(r,h_j),$$

as $\deg(g_3-g_j)=n$, for $j\in\mathcal{J}$, where $S(r)$ is a defined as in Lemma 2.8, and this is not possible.

Next, we consider $b_3=-c_1\beta_1-c_0$. i.e., $\beta_1=-\frac{c_0}{c_1}$, a constant, which in turn implies β is a polynomial with $\deg(\beta)=n=1$. Then γ is also a polynomial with $\deg(\gamma)=n=1$, and hence β_1 is a constant. We suppose that

$$\deg(\gamma-2\beta)=n=1,\ \ \deg(\gamma+\beta)=n=1.$$

Then one can get that $\deg(g_6-g_j)=0$, for $j=0,1,2,4,5,6$.

Then replacing g_3 by g_6, and applying Lemma 2.8, as discussed above, we can get a contradiction. Next, we assume that either $\gamma-2\beta$ or $\gamma+\beta$ is constant, say $\mathcal{A}$. So, we discuss the following two cases in our discussions.

If $\gamma-2\beta$ is constant, then we see that the functions g_j takes the form

$$\begin{aligned}
g_0&=2\gamma=4\beta+2\mathcal{A};\\
g_1&=5\beta+2\mathcal{A};\\
g_2&=\beta+\gamma=2\beta+\mathcal{A};\\
g_3&=2\beta;\\
g_4&=2\beta+\gamma=4\beta+\mathcal{A};\\
g_5&=\beta;\\
g_6&=\gamma=2\beta+\mathcal{A}.
\end{aligned}$$

Since β_1 and γ_1 both are constants, so we see that all b_j are constants. So keeping $b_3=0$, in our mind, the identity (3.25) becomes

$$t_5e^{5\beta}+t_4e^{4\beta}+t_2e^{2\beta}+t_1e^{\beta}\equiv 0, \tag{3.26}$$

where $t_5=b_0e^{2\mathcal{A}}$, $t_4=b_0e^{2\mathcal{A}}+b_4e^{\mathcal{A}}$, $t_2=b_2e^{\mathcal{A}}+b_6e^{\mathcal{A}}$, $t_1=b_5$.

The identity (3.26) shows that $b_1=0$, $b_0e^{\mathcal{A}}+b_4=0$, $b_2+b_6=0$ and$b_5=0$, which in turn implies that

$$\gamma_1 \equiv e^{\gamma(z+c)+\gamma(z)} = 0;$$
$$e^{\gamma(z+c)-\gamma(z)}e^{\mathcal{A}} + c_1 e^{\beta(z+c)-\beta(z)} + c_0 e^{\gamma(z+c)-\gamma(z)} \equiv 0;$$
$$\left\{e^{\gamma(z+c)-\gamma(z)} + 1\right\}\frac{b-a(c_0+c_1)}{b-a} - (c_0+1)e^{\gamma(z+c)-\gamma(z)} - c_1 - 1 \equiv 0;$$
$$c_1 + c_0 + 1 - \frac{b-a(c_0+c_1)}{b-a} \equiv 0.$$

This is impossible since $a \neq b$.

Next we suppose that $\beta + \gamma$ is constant. Proceeding exactly the same as done above, we can arrive a contradiction.

This completes the proof of *Theorem 1.2*. □

References

[1] Bergweiler, W., Langley, J.K.: Zeros of difference of meromorphic functions. Math. Proc. Camb. Philos. Soc. **142**, 133–147 (2007)

[2] Brück, R.: On entire functions which share one value ***CM*** with their derivative. Results Math. **30**, 21–24 (1996)

[3] Chen, Z.X.: Some results on difference Riccati equations. Acta Math. Sin. Eng. Ser. **27**, 1091–1100 (2011)

[4] Chen, Z.X., Yi, H.X.: On sharing values of meromorphic functions and their differences. Results Math. **63**, 557–565 (2013)

[5] Chiang, Y.M., Feng, S.J.: On the Nevanlinna characteristic of $f(z + c)$ and difference equation in the complex plane. Ramanujan J. **16**, 105–129 (2008)

[6] Gundersen, G.: Meromorphic function share four values. Trans. Am. Math. Soc. **277**, 545–567 (1983)

[7] Gundersen, G.: Correction to meromorphic functions that share four values. Trans. Am. Math. Soc. **304**, 847–850 (1987)

[8] Halburd, R.G., Korhonen, R.J.: Nevanlinna theory for the difference operator. Ann. Acad. Sci. Fenn. Math. **31**, 463–478 (2006)

[9] Halburd, R.G., Korhonen, R.J.: Meromorphic solution of difference equations, integrability and the discrete. J. Phys. A **40**, 1–38 (2007)

[10] Hayman, W.K.: Meromorphic Functions. Clarendon Press, Oxford (1964)

[11] Heittokangas, J., Korhonen, R., Laine, I., et al.: Value sharing results for shifts of meromorphic functions, and sufficient condition for periodicity. J. Math. Anal. Appl. **355**, 352–363 (2009)

[12] Liu, K.: Zeros of difference polynomials of meromorphic functions. Results Math. **57**, 365–376 (2010)

[13] Liu, K., Yang, L.Z.: Value distribution of the difference operator. Arch. Math. **92**, 270–278 (2009)

[14] Lü, F., Lü, W.: Meromorphic functions sharing three values with their difference operators. Comput. Methods Funct. Theory. (2017). https://doi.org/10.1007/s40315-016-0188-5

[15] Mohonko, A.Z.: The Nevanlinna characteristics of certain meromorphic functions. Teor. Funktsii Funktsional. Anal. i Prilozhen **14**, 83–87 (1971). **(Russian)**

[16] Mues, E.: Meromorphic functions sharing four values. Complex Var. Theory Appl. **12**, 169–179 (1989)

[17] Nevanlinna, R.: Le Théoréme de Picard-Borel et la Théorie des Fonctions Méromorphes. Gauthier-Villars, Paris (1929)

[18] Rubel, L., Yang, C.C.: Values shared by an entire function and its derivative. In: Lecture Notes in Mathematics, vol. 599, pp. 101–103. Springer, Berlin (1977)

[19] Valiron, G.: Sur la dérivée des fonctions algébroïdes. Bull. Soc. Math. France **59**, 17–39 (1931)

[20] Yang, C.C., Yi, H.X.: Uniqueness Theory of Meromorphic Functions. Kluwer Academic Publishing Group, Dordrecht (2003)

[21] Yang, C.C., Yi, H.X.: Uniqueness Theory of Meromorphic Functions. Science Press, Beijing (2006)

[22] Yang, L.: Value Distribution Theory. Springer, Berlin (1993)

[23] Zhang, J., Liao, L.W.: Entire function sharing some values with their difference operators. Sci. China Math. **57**(10), 2143–2152 (2014)

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