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REAL AND COMPLEX ANALYSIS

# Results on Meromorphic Function Sharing Two Sets with Its Linear $c$-Difference Operator

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**Abstract**—In this paper, the existing results concerning difference operator sharing two sets have been extended up to the most general form, namely linear difference operator. Furthermore, we have been able to find out the specific form of the function. A considerable number of examples have been exhibited throughout the paper pertinent with different issues.

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## 1. INTRODUCTION, DEFINITIONS, AND RESULTS

Through out the paper, the term "meromorphic" (resp., "entire") will always mean meromorphic in the whole complex plane $\mathbb{C}$ which are non-constant, unless specifically stated otherwise. We shall adopt the standard notations of the Nevanlinna's value distribution theory of meromorphic functions from ([9, 16]). For such a meromorphic function $f$ and $a \in \overline{\mathbb{C}} =: \mathbb{C} \cup \{\infty\}$, each $z$ with $f(z) = a$ will be called $a$-point of $f$. We denote $\mathbb{C}^*$ by $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

In 1926, Nevanlinna first showed that a non-constant meromorphic function on the complex plane $\mathbb{C}$ is uniquely determined by the pre-images, ignoring multiplicities, of five distinct values (including infinity). The beauty of this result lies in the fact that there is no counterpart of this result in the real function theory. A few years latter, he showed that when multiplicities are taken into consideration, four points are enough and in that case either the two functions coincides or one is the bilinear transformation of the other one. Clearly these results initiated the study of uniqueness of two meromorphic functions $f$ and $g$. The study becomes more interesting if the function $g$ is related with $f$.

If for $a \in \mathbb{C} \cup \{\infty\}$, $f$ and $g$ have the same set of $a$-points with same multiplicities then we say that $f$ and $g$ share the value $a$ $CM$ (Counting Multiplicities). If we do not take the multiplicities into account, $f$ and $g$ are said to share the value an $IM$ (Ignoring Multiplicities).

**Definition 1.1.** *For a non-constant meromorphic function $f$ and any set $\mathcal{S} \subset \overline{\mathbb{C}}$, we define*

$$E_f(\mathcal{S}) = \bigcup_{a\in\mathcal{S}} \Big\{(z,p) \in \mathbb{C} \times \mathbb{N} : f(z) = a,\ with\ multiplicity\ p\Big\},$$

$$\overline{E}_f(\mathcal{S}) = \bigcup_{a\in\mathcal{S}} \Big\{(z,1) \in \mathbb{C} \times \{1\} : f(z) = a\Big\}.$$

If $E_f(\mathcal{S}) = E_g(\mathcal{S})$ ($\overline{E}_f(\mathcal{S}) = \overline{E}_g(\mathcal{S})$), then we simply say $f$ and $g$ share $\mathcal{S}$ Counting Multiplicities (CM) (Ignoring Multiplicities (IM)).

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More formally it can be explained as follows.

**Definition 1.2.** [3] *If $f$ is a meromorphic function and $S \subset \overline{\mathbb{C}}$ then if $z_0 \in f^{-1}(S)$, the value of $E_f(S)$ at the point $z_0$ is denoted by $E_f(S)(z_0) : f^{-1}(S) \to \mathbb{N}$ and is equal to the multiplicity of zero of the function $f(z) - f(z_0)$ at $z_0$ i.e. the order of the pole of the function $(f(z) - f(z_0))^{-1}$ at $z_0$ if $f(z_0) \in \mathbb{C}$ (resp. of the function $f(z)$ if $z_0$ is a pole for $f$).*

Evidently, if $\mathcal{S}$ contains one element only, then it coincides with the usual definition of $CM(IM)$ sharing of values.

In 2001, an idea of gradation of sharing known as weighted sharing has been introduced by *Lahiri* [11, 12] which measure how close a shared value is to being share $CM$ or to being shared $IM$. So for the purpose of relaxing the nature of sharing the sets, the notion of weighted sharing of values and sets, has become an effective tool.

Recently, the definition have been reorganized by us [3] as follows.

**Definition 1.3.** [3] *For $k \in \overline{\mathbb{N}}$ and $z_0 \in f^{-1}(S)$, let us put that $E_f(S,k)(z_0) = \min\{E_f(S)(z_0), k+1\}$. Given $S \subset \overline{\mathbb{C}}$, we say that meromorphic functions $f$ and $g$ share the set $S$ up to multiplicity $k$ (or share $S$ with weight $k$, or simply share $(S,k)$) if $f^{-1}(S) = g^{-1}(S)$ and for each $z_0 \in f^{-1}(S)$ we have $E_f(S,k)(z_0) = E_g(S,k)(z_0)$, which is represented by the notation $E_f(S,k) = E_g(S,k)$.*

As we proceed through the literature of the shift and difference operators of a meromorphic function $f$, we feel that there should be a streamline in the definitions. This is one of the motivations of writing this paper. To this end, below we are providing several definitions in a compact and convenient way.

In what follows, $c$ always means a non-zero constant. We now define the shift and difference operator in the following manner.

**Definition 1.4.** *For a meromorphic function $f$, let us now denote its shift $I_cf$ and difference operators $\Delta_c f$ respectively by $I_cf(z) = f(z+c)$ and $\Delta_c f(z) = (I_c - 1)f(z) = f(z+c) - f(z)$.*

*Next we define $\Delta_c^s f := \Delta_c^{s-1}(\Delta_c f), \forall s \in \mathbb{N} - \{1\}$.*

For the purpose of generalizing the above definitions, we now propose the definition of linear shift operator $\mathcal{L}_p(f,I)$ as follows.

**Definition 1.5.** *For a meromorphic function $f$ and a positive integer $p$, we define*

$$\begin{aligned} \mathcal{L}_p(f,I) &= a_p I_{c_p} f(z) + a_{p-1} I_{c_{p-1}} f(z) + \ldots + a_0 I_{c_0} f(z) \\ &= a_p f(z+c_p) + \ldots + a_1 f(z+c_1) + a_0 f(z+c_0), \end{aligned} \tag{1.1}$$

$a_p(\neq 0), \ldots, a_1, a_0 \in \mathbb{C}$, $c_p, \ldots, c_1, c_0 \in \mathbb{C}$.

In particular, for suitable choice of $c_j$, say $c_j = jc$, for $j \in \{0,1,\ldots,p\}$, we call $\mathcal{L}_p(f,I)$ as a linear $c$-shift operator $\mathcal{L}_p(f,I_c)$ as follows.

**Definition 1.6.** *For $c \in \mathbb{C}^*$ and a positive integer $p$, we define*

$$\begin{aligned} \mathcal{L}_p(f,I_c) &= a_p I_{pc} f(z) + a_{p-1} I_{(p-1)c} f(z) + \ldots + a_0 I_0 f(z) \\ &= a_p f(z+pc) + a_{p-1} f(z+(p-1)c) + \ldots + a_0 f(z). \end{aligned} \tag{1.2}$$

Analogous to the definitions 1.5 and 1.6, we now introduce the definitions of linear difference operator $\mathcal{L}_p(f,\Delta)$ and linear $c$-difference operator $\mathcal{L}_p(f,\Delta_c)$ in the following manner.

**Definition 1.7.**

$$\begin{aligned} \mathcal{L}_p(f,\Delta) &= a_p \Delta_{c_p} f(z) + a_{p-1} \Delta_{c_{p-1}} f(z) + \ldots + a_0 \Delta_{c_0} f(z) = a_p f(z+c_p) + \ldots + a_1 f(z+c_1) \\ &+ a_0 f(z+c_0) - \left(\sum_{j=0}^{p} a_j\right) f(z) = \mathcal{L}_p(f,I) - \left(\sum_{j=0}^{p} a_j\right) f(z), \end{aligned} \tag{1.3}$$

**Definition 1.8.** *For $c \in \mathbb{C}^*$, a positive integer $p$, putting $c_j = jc$, $j \in \{0,1,\ldots,p\}$, in (1.3) we define*

$$\mathcal{L}_p(f,\Delta_c) = a_p \Delta_{pc} f(z) + a_{p-1} \Delta_{(p-1)c} f(z) + \ldots$$

$$+a_1\Delta_c f(z)+a_0\Delta_0 f(z)=\mathcal{L}_p(f,I_c)-\left(\sum_{j=0}^{p}a_j\right)f(z). \tag{1.4}$$

For the specific choices of the constants as $a_j=(-1)^{p-j}\binom{p}{j}$, where $0\leqslant j\leqslant p$, in the expression $\mathcal{L}_p(f,\Delta_c)$, one can easily get that $\mathcal{L}_p(f,\Delta_c)=\Delta_c^p f$.

For the sake of convenience, we are now going to introduce linear $c$-difference odd operator $\mathcal{L}_p^o(f,\Delta_c)$ as follows:

**Definition 1.9.** *For $c\in\mathbb{C}^*$, putting $c_j=(2j+1)c$, $j\in\{0,1,\ldots,p\}$, in (1.3) we define,*

$$\mathcal{L}_p^o(f,\Delta_c)=a_p\Delta_{(2p+1)c}f(z)+a_{p-1}\Delta_{(2p-1)c}f(z)+\ldots+a_1\Delta_1 f(z)+a_0\Delta_c f(z). \tag{1.5}$$

Henceforth unless otherwise stated for $a\neq 0$, throughout the paper, we denote, for $n\in\mathbb{N}$, by $\mathcal{S}_a^n=\{a,a\theta,a\theta^2,\ldots,a\theta^{n-1}\}$, where $\theta=\exp\left(\frac{2\pi i}{n}\right)$, and $S_2=\{\infty\}$.

Recently a number of papers ([6, 8, 21] etc.) have focused on the value distribution in difference analogues of meromorphic functions.

In this perspective, many researchers have become interested to deal with the uniqueness problem of meromorphic function that share values or sets with its shift or difference operators. Below we are mentioning few of them.

**Theorem A** [21]. *Let $c\in\mathbb{C}^*$, and suppose that $f(z)$ is a non-constant meromorphic function with finite order such that $E_f(\mathcal{S}_1^n,\infty)=E_{I_cf}(\mathcal{S}_1^n,\infty)$ and $E_f(\mathcal{S}_2,\infty)=E_{I_cf}(\mathcal{S}_2,\infty)$. If $n\geqslant 4$, then $I_cf\equiv tf$, where $t^n=1$.*

The following example shows that Theorem A is not valid for 'infinite ordered' meromorphic function.

**Example 1.1.** *Let $c\in\mathbb{C}^*$ and $f(z)=\exp\left(\sin\left(\frac{\pi z}{c}\right)\right)$. It is clear that $I_cf=\exp\left(-\sin\left(\frac{\pi z}{c}\right)\right)$. It is easy to verify that $E_f(\mathcal{S}_1^n,\infty)=E_{I_cf}(\mathcal{S}_1^n,\infty)$ and $E_f(\mathcal{S}_2,\infty)=E_{I_cf}(\mathcal{S}_2,\infty)$ for any value of $n\in\mathbb{N}$ but the conclusion of Theorem A ceases to hold.*

**Example 1.2.** *Let $c\in\mathbb{C}^*$ and $f(z)=\exp\left(\exp\left(\frac{\pi iz}{c}\right)\right)$. It is clear that $I_cf=\exp\left(-\exp\left(\frac{\pi iz}{c}\right)\right)$. It is easy to verify that $E_f(\mathcal{S}_1^n,\infty)=E_{I_cf}(\mathcal{S}_1^n,\infty)$ and $E_f(\mathcal{S}_2,\infty)=E_{I_cf}(\mathcal{S}_2,\infty)$ for any value of $n\in\mathbb{N}$ but the conclusion of Theorem A ceases to hold.*

The next examples show that for $n=1$ or $n=2$ Theorem A is not true.

**Example 1.3.** *Let $f(z)=\dfrac{e^{\mathcal{B}z}+\sin^2\left(\frac{2\pi z}{c}\right)-1}{\sin^2\left(\frac{2\pi z}{c}\right)-1}$, where $e^{\mathcal{B}c}=-1$. It easy to verify that $E_f(\mathcal{S}_1^1,\infty)=E_{I_cf}(\mathcal{S}_1^1,\infty)$ and $E_f(\mathcal{S}_2,\infty)=E_{I_cf}(\mathcal{S}_2,\infty)$ but $I_cf\not\equiv f$.*

**Example 1.4.** *Let $f(z)=\dfrac{\exp\left(\frac{\pi iz}{2c}\right)-\exp\left(-\frac{\pi iz}{2c}\right)a^2}{\sqrt{2}ia}$, where $a$ is a non-zero constant. It easy to verify that $E_f(\mathcal{S}_1^2,\infty)=E_{I_cf}(\mathcal{S}_1^2,\infty)$ and $E_f(\mathcal{S}_2,\infty)=E_{I_cf}(\mathcal{S}_2,\infty)$ but $I_cf\not\equiv f$.*

By replacing $I_cf$ by $\Delta_cf$ in *Theorem A*, *Chen–Chen* [5] obtained the following result.

**Theorem B** [5]. *Let $c\in\mathbb{C}^*$ and $\mathcal{S}_a^n$ and $\mathcal{S}_2$ be defined as in* Theorem A. *Suppose that $f(z)$ is a non-constant meromorphic function with finite order such that $E_f(\mathcal{S}_a^n,2)=E_{\Delta_cf}(\mathcal{S}_a^n,2)$ and $E_f(\mathcal{S}_2,\infty)=E_{\Delta_cf}(\mathcal{S}_2,\infty)$. If $n\geqslant 7$, then $\Delta_cf\equiv tf$, where $t^n=1$ with $t\neq -1$.*

In this direction, Banerjee–Bhattacharyya [4] successfully reduced the weight of the sets as well as the lower bound of $n$ in Theorem B, by obtaining the following two results.

**Theorem C** [4]. *Suppose that $f$ is a non-constant meromorphic function of finite order such that $E_f(\mathcal{S}_b^n,2)=E_{\Delta_cf}(\mathcal{S}_b^n,2)$, where $b^n=a\in\mathbb{C}^*$ and $E_f(\mathcal{S}_2,0)=E_{\Delta_cf}(\mathcal{S}_2,0)$, and $n\geqslant 6$. Then there is a constant $t\in\mathbb{C}$ such that $\Delta_cf\equiv tf$, where $t^n=1$ and $t\neq -1$.*

**Theorem D.** *Suppose that $f$ is a non-constant meromorphic function of finite order, $\mathcal{S}_b^n$ be defined as in Theorem C, and such that $E_f(\mathcal{S}_b^n, 1) = E_{\Delta_c f}(\mathcal{S}_b^n, 1)$ and $E_f(\mathcal{S}_2, 0) = E_{\Delta_c f}(\mathcal{S}_2, 0)$, and $n \geqslant 7$. Then there is a constant $t \in \mathbb{C}$ such that $\Delta_c f \equiv tf$, where $t^n = 1$ and $t \neq -1$.*

The following examples show that the condition 'finite orderedness' of the function $f$ is not necessary in Theorems B, C, D.

**Example 1.5.** *For a complex number $t(\neq -1)$, let*

$$f(z) = \frac{\exp\left(\frac{z}{c}\log(t^{\frac{1}{p}}+1)\right)}{\exp\left(\exp\left(\dfrac{2\pi iz}{c}\right)\right)-1}$$

*It is easy to verify that $\Delta_c^p f \equiv tf$, for all positive integer $p$. As $t$ is a complex constant satisfying $t^n = 1$, it follows that $(\Delta_c^p f)^n - 1 \equiv f^n - 1$. Hence $E_{\Delta_c^p f}(\mathcal{S}_1^n, \infty) = E_f(\mathcal{S}_1^n, \infty)$ and $E_{\Delta_c^p f}(\mathcal{S}_2, \infty) = E_f(\mathcal{S}_2, \infty)$.*

In the same manner more examples can be formed as follows:

**Example 1.6.** *Let* $f(z) = \dfrac{\exp\left(\frac{z}{c}\log(t^{\frac{1}{p}}+1)\right)\sin\left(\frac{2\pi z}{c}\right)}{\exp\left(\sin\left(\frac{2\pi z}{c}\right)\right)-1}$.

**Example 1.7.** *Let* $f(z) = \dfrac{\exp\left(\frac{z}{c}\log(t^{\frac{1}{p}}+1)\right)\cos\left(\frac{2\pi z}{c}\right)}{\exp\left(\cos\left(\frac{2\pi z}{c}\right)\right)-1}$.

**Example 1.8.** *Let* $f(z) = \dfrac{\exp\left(\frac{z}{c}\log(t^{\frac{1}{p}}+1)\right)\exp\left(\frac{2k\pi iz}{c}\right)}{\exp\left(\exp\left(\dfrac{2\pi iz}{c}\right)\right)-1}$.

Recently, in this direction *Deng−Liu−Yang* [7] obtained the following result.

**Theorem E** [7]. *Let $c \in \mathbb{C}^*$ and $\mathcal{S}_a^n$, $\mathcal{S}_2$ be defined as in Theorem A. Suppose that $f(z)$ is a non-constant meromorphic function such that $E_f(\mathcal{S}_a^n, k) = E_{\Delta_c f}(\mathcal{S}_a^n, k)$ and $E_f(\mathcal{S}_2, \infty) = E_{\Delta_c f}(\mathcal{S}_2, \infty)$. If $n \geqslant 7$, when $k = 1$ or $n \geqslant 5$, when $k \geqslant 2$, then $\Delta_c f \equiv tf$, where $t^n = 1$ with $t \neq -1$.*

**Remark 1.1.** We know that all the lemmas and hence the corresponding results so far obtained based on the lemmas related to a function and its shift $I_c f$ or $\Delta_c f$ are for finite ordered meromorphic functions only, so we have a strong doubt about the validity of Theorem E for the case of "infinite ordered" meromorphic function.

For the purpose of further improvements as well as extensions of Theorems B, C, D, E, we propose the following questions.

(i) Can we replace the difference operator $\Delta_c f$ by a more general setting $\mathcal{L}_p(f, \Delta_c)$ in Theorem B, C, D, E?

(ii) Is it possible to relax the nature of sharing $(\mathcal{S}_2, \infty)$ in Theorems B, E further by $(\mathcal{S}_2, 0)$?

In this paper, we have answered the above questions affirmatively. Followings are the main result of this paper.

**Theorem 1.1.** *Let $n, p \in \mathbb{N}$, and $f$ be a non-constant meromorphic function of finite order such that $E_f(\mathcal{S}_a^n, 1) = E_{\mathcal{L}_p(f,\Delta_c)}(\mathcal{S}_a^n, 1)$ and $E_f(\mathcal{S}_2, 0) = E_{\mathcal{L}_p(f,\Delta_c)}(\mathcal{S}_2, 0)$. If $n \geqslant \max\left\{p+4, 7\right\}$, then there exists a constant $t \in \mathbb{C}$ such that $\mathcal{L}_p(f, \Delta_c) \equiv tf$, where $t^n = 1$ and $t \neq -1$.*

**Theorem 1.2.** *Let $n, p \in \mathbb{N}$, and $f$ be a non-constant meromorphic function of finite order such that $E_f(\mathcal{S}_a^n, 2) = E_{\mathcal{L}_p(f,\Delta_c)}(\mathcal{S}_a^n, 2)$ and $E_f(\mathcal{S}_2, 0) = E_{\mathcal{L}_p(f,\Delta_c)}(\mathcal{S}_2, 0)$. If $n \geqslant \max\left\{p+3, 6\right\}$, then the conclusion of Theorem 1.1 holds.*

**Remark 1.2.** Since $I_c f$, $\Delta_c f$ and $\mathcal{L}_p(f, I_c)$ are the very special forms of $\mathcal{L}_p(f, \Delta_c)$, so it is clear that Theorem 1.1 and Theorem 1.2 improved and extended the Theorems B, C, D and E in a large extent.

Let us denote by $\mathbb{P}_c$ as the field of periods $c(\neq 0)$ of meromorphic functions defined in $\mathbb{C}$. That means

$$\mathbb{P}_c = \{g : g \text{is meromorphic and} g(z+c) = g(z), \forall z \in \mathbb{C}\}.$$

From *Theorem 1.1* and Theorem 1.2, we can now easily deduce the following Corollaries:

**Corollary 1.1.** *Let $n, s \in \mathbb{N}$, and $f$ be a non-constant meromorphic function of finite order such that $E_f(\mathcal{S}_1^n, 1) = E_{\Delta_c^s f}(\mathcal{S}_1^n, 1)$ and $E_f(\mathcal{S}_2, 0) = E_{\Delta_c^s f}(\mathcal{S}_2, 0)$. If $n \geqslant \max\left\{s+4, 7\right\}$, then there exists a constant $t \in \mathbb{C}$ such that $\Delta_c^s f \equiv tf$, where $t^n = 1$ and $t \neq -1$.*

**Corollary 1.2.** *Let $n, s \in \mathbb{N}$, and $f$ be a non-constant meromorphic function of finite order such that $E_f(\mathcal{S}_1^n, 2) = E_{\Delta_c^s f}(\mathcal{S}_1^n, 2)$ and $E_f(\mathcal{S}_2, 0) = E_{\Delta_c^s f}(\mathcal{S}_2, 0)$.* If $n \geqslant \max\left\{s+3, 6\right\}$, *then the conclusion of Corollary 1.1 holds.*

**Remark 1.3.** From Examples 1.1 and 1.2, we see that Corollaries 1.1 and 1.2 are not valid for 'infinite ordered' meromorphic functions for the case $s = 1$, $a_1 = 1$, $a_0 = 0$.

**Corollary 1.3.** *Let $s$, where $1 \leqslant s \leqslant 3$, be an integer and $f$ be a non-constant meromorphic function of finite order. Suppose $E_f(\mathcal{S}_1^7, 1) = E_{\Delta_c^s f}(\mathcal{S}_1^7, 1)$ and $E_f(\mathcal{S}_2, 0) = E_{\Delta_c^s f}(\mathcal{S}_2, 0)$. Then there exists a constant $t \in \mathbb{C}$ such that $\Delta_c^s f \equiv tf$, where $t^7 = 1$ and $t \neq -1$.*

**Corollary 1.4.** *Let $s$, where $1 \leqslant s \leqslant 3$, be an integer and $f$ be a non-constant meromorphic function of finite order. Suppose $E_f(\mathcal{S}_1^6, 1) = E_{\Delta_c^s f}(\mathcal{S}_1^6, 1)$ and $E_f(\mathcal{S}_2, 0) = E_{\Delta_c^s f}(\mathcal{S}_2, 0)$. Then there exists a constant $t \in \mathbb{C}$ such that $\Delta_c^s f \equiv tf$, where $t^6 = 1$ and $t \neq -1$.*

From the following three examples we see that the conclusion of Corollary 1.3 and Corollary 1.4 actually occurs for the case $s = 1$, $s = 2$ and $s = 3$.

**Example 1.9.** *Let* $f(z) = (1+\zeta)^{z/c} \dfrac{\exp\left(\frac{2\pi i z}{c}\right)}{\exp\left(\frac{2\pi i z}{c}\right) - 1}$, *where* $\zeta = \exp\left(\dfrac{2\pi i}{7}\right)$ $\left(\zeta = \exp\left(\dfrac{2\pi i}{6}\right)\right)$. *Clearly* $E_f(\mathcal{S}_1^7, 1) = E_{\Delta_c f}(\mathcal{S}_1^7, 1)$ $(E_f(\mathcal{S}_1^6, 2) = E_{\Delta_c f}(\mathcal{S}_1^6, 2))$ *and* $E_f(\mathcal{S}_2, 0) = E_{\Delta_c f}(\mathcal{S}_2, 0)$ *and* $\Delta_c f \equiv \zeta f$.

**Example 1.10.** *Let* $f(z) = \left(1+\sqrt{\zeta}\right)^{z/c} \dfrac{\sin\left(\frac{2\pi z}{c}\right)}{\sin\left(\frac{2\pi z}{c}\right) - 1}$, *where* $\zeta = \exp\left(\dfrac{2\pi i}{7}\right) \times \left(\zeta = \exp\left(\dfrac{2\pi i}{6}\right)\right)$. *Clearly* $E_f(\mathcal{S}_1^7, 1) = E_{\Delta_c f}(\mathcal{S}_1^7, 1)$ $(E_f(\mathcal{S}_1^6, 2) = E_{\Delta_c f}(\mathcal{S}_1^6, 2))$ *and* $E_f(\mathcal{S}_2, 0) = E_{\Delta_c^2 f}(\mathcal{S}_2, 0)$ *and* $\Delta_c^2 f \equiv \zeta f$.

**Example 1.11.** *Let* $f(z) = \left(1+\sqrt[3]{\zeta}\omega\right)^{z/c} \dfrac{\cos\left(\frac{2\pi z}{c}\right)}{\exp\left(\frac{2\pi i z}{c}\right) - 1}$, *where* $\zeta = \exp\left(\dfrac{2\pi i}{7}\right) \times \left(\zeta = \exp\left(\dfrac{2\pi i}{6}\right)\right)$. *Clearly* $E_f(\mathcal{S}_1^7, 1) = E_{\Delta_c f}(\mathcal{S}_1^7, 1)$ $(E_f(\mathcal{S}_1^6, 2) = E_{\Delta_c f}(\mathcal{S}_1^6, 2))$ *and* $E_f(\mathcal{S}_2, 0) = E_{\Delta_c^3 f}(\mathcal{S}_2, 0)$ *and* $\Delta_c^3 f \equiv \zeta f$.

**Remark 1.4.** We note that the linear difference equation

$$\Delta_c^s f(z) = \sum_{i=0}^{s} (-1)^{s-i} \binom{n}{i} f(z+ci) = tf(z), \tag{1.6}$$

where $t^s = 1, t \neq -1$, can be solved in terms of linear combinations of exponential functions with coefficients in $\mathbb{P}_c$. In fact, if $f$ be a finite ordered meromorphic function satisfies the relation $\Delta_c^s f \equiv tf$, then $f(z)$ must assume the following form

$$f(z) = \pi_{s-1}(z)\alpha_{s-1}^{\frac{z}{c}} + \ldots + \pi_0(z)\alpha_0^{\frac{z}{c}},$$

where all $\pi_j \in \mathbb{P}_c$, and $\alpha_j$ are roots of the equation $\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}z^j = t$.

Following example shows that in Theorems 1.1 and 1.2 the term 'finite order meromorphic functions' can not be removed for a special class of linear $c$-difference odd operator, where $a_j = (-1)^j\binom{p}{j}2^{p-j}$. We note that in this case (1.5) takes the form $\mathcal{L}_p^o(f,\Delta_c) = \sum_{j=0}^{p} a_j f(z+(2j+1)c)$.

**Example 1.12.** *For $c \in \mathbb{C}^*$, we suppose that $f(z) = \exp\left(\cos\left(\frac{\pi z}{c}\right)\right)$. We choose $\mathcal{L}_p(f,\Delta_c)$ as $\mathcal{L}_p^o(f,\Delta_c)$. Since $\cos\left(\frac{\pi(z+(2j+1)c)}{c}\right) = -\cos\left(\frac{\pi z}{c}\right)$ and it follows that $\mathcal{L}_p^o(f,\Delta_c) = \exp\left(-\cos\left(\frac{\pi z}{c}\right)\right)$; so $f$ satisfies all the conditions of Theorems 1.1 and 1.2 but $\mathcal{L}_p^o(f,\Delta_c) \not\equiv tf$.*

However, unfortunately, we were not succeeded to find any counter example for general linear $c$-difference operator.

The next example shows that the set $\mathcal{S}_1$ in Corollary 1.3 simply can not be replaced by an arbitrary set.

**Example 1.13.** *Let $\mathcal{S}_a^{\#} = \left\{a, \frac{a}{\sqrt{\omega}}, \frac{a}{\omega}, 0, \frac{a}{\omega\sqrt{\omega}}, a\omega, a\sqrt{\omega}\right\}$ and $\mathcal{S}_2 = \{\infty\}$, where $a$ is any non-zero complex number, $\omega$ is non-real cube root of unity,*

$$f(z) = \exp\left(\frac{z}{c}\log(\omega^{\frac{1}{2p}}+1)\right)\frac{1}{\cos^2\left(\frac{2\pi z}{c}\right)-1},$$

*where $p (1 \leqslant p \leqslant 4)$ be an integer.*

It is easy to verify that $E_f(\mathcal{S}_a^{\#},1) = E_{\Delta_c^p f}(\mathcal{S}_a^{\#},1)$ and $E_f(\mathcal{S}_2,0) = E_{\Delta_c^p f}(\mathcal{S}_2,0)$ but neither $\Delta_c^p f \equiv f$ with $t^7 = 1$ nor $f$ has the specific form as above.

Though the standard definitions and notations of the value distribution theory are available in [9, 16], we explain here some of them which are used in the paper.

**Definition 1.10** [13]. *For $a \in \mathbb{C}\cup\{\infty\}$, we denote by $N(r,a;f\mid=1)$ the counting function of simple $a$-points of $f$. For a positive integer $p$, we denote $N(r,a;f\mid\leqslant p)(N(r,a;f\mid\geqslant p))$ the counting function of those $a$-points of $f$ whose multiplicities are not greater (less) than $p$ where each $a$-point is counted according to its multiplicity.*

*$\overline{N}(r,a;f\mid\leqslant p)(\overline{N}(r,a;f\mid\geqslant p))$ are defined similarly, where in counting the $a$-points of $f$ we ignore the multiplicities.*

**Definition 1.11** [11]. *We denote by $N_2(r,a;f)$ the sum $\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geqslant 2)$.*

**Definition 1.12** [18, 20]. *Let $f$ and $g$ be two non-constant meromorphic functions such that $f$ and $g$ share the value 1 $IM$. Let $z_0$ be a 1-point of $f$ with multiplicity $p$, a 1-point of $g$ with multiplicity $q$. We denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of $f$ and $g$ where $p > q$, each point in this counting function is counted only once. In the same way we can define $\overline{N}_L(r,1;g)$.*

**Definition 1.13** [6, 9]. *Let $f, g$ share a value $IM$. We denote by $\overline{N}_*(r,a;f,g) = \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.*

**Definition 1.14** [14]. *Let $a,b \in \mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f|g=b)$ the counting function of those $a$-points of $f$, counted according to multiplicity, which are $b$-points of $g$.*

## 2. SOME USEFUL LEMMAS

In this section, we are going to discuss some lemmas which will needed later to prove our main results. We define, for a non-constant meromorphic functions $f$,

$$\mathcal{F}=\left(\frac{f}{a}\right)^n,\quad \mathcal{G}=\left(\frac{\mathcal{L}_p(f,\Delta_c)}{a}\right)^n. \tag{2.1}$$

Associated to $\mathcal{F}$ and $\mathcal{G}$, we next define $\mathcal{H}$ and $\Psi$ as follows:

$$\mathcal{H}=\left(\frac{\mathcal{F}''}{\mathcal{F}'}-\frac{2\mathcal{F}'}{\mathcal{F}-1}\right)-\left(\frac{\mathcal{G}''}{\mathcal{G}'}-\frac{2\mathcal{G}'}{\mathcal{G}-1}\right), \tag{2.2}$$

and

$$\Psi=\frac{\mathcal{F}'}{\mathcal{F}(\mathcal{F}-1)}-\frac{\mathcal{G}'}{\mathcal{G}(\mathcal{G}-1)}. \tag{2.3}$$

**Lemma 2.1.** [6] *Let $g$ be a meromorphic function of finite order $\rho$, and let $c\in\mathbb{C}^*$ be fixed. Then for each $\epsilon>0$, we have*

$$m\left(r,\frac{g(z+c)}{g(z)}\right)+m\left(r,\frac{g(z)}{g(z+c)}\right)=O(r^{\rho-1+\epsilon}).$$

**Lemma 2.2.** *Let $\mathcal{F}$ and $\mathcal{G}$ be given by (2.1) satisfying $E_{\mathcal{F}}(1,m)=E_{\mathcal{G}}(1,m)$, $0\leqslant m<\infty$ with $\mathcal{H}\not\equiv 0$, then*

$$N_E^{1)}\left(r,\frac{1}{\mathcal{F}-1}\right)=N_E^{1)}\left(r,\frac{1}{\mathcal{G}-1}\right)\leqslant N(r,\mathcal{H})+S(r,\mathcal{F})+S(r,\mathcal{G}).$$

**Proof.** Since $E_{\mathcal{F}}(1,q)=E_{\mathcal{G}}(1,q)$, so it is obvious that any simple 1-point of $\mathcal{F}$ and $\mathcal{G}$ is a zero of $\mathcal{H}$. The construction of $\mathcal{H}$ implies that, $m(r,\mathcal{H})=S(r,\mathcal{F})+S(r,\mathcal{G})$. By the *First Fundamental Theorem*, we get

$$N_E^{1)}\left(r,\frac{1}{\mathcal{F}-1}\right)=N_E^{1)}\left(r,\frac{1}{\mathcal{G}-1}\right)\leqslant N\left(r,\frac{1}{\mathcal{H}}\right)\leqslant N(r,\mathcal{H})+S(r,\mathcal{F})+S(r,\mathcal{G}).$$

The proof is complete. □

**Lemma 2.3** [10]. *Let $f$ be a non-constant meromorphic function of finite order and $c\in\mathbb{C}^*$. Then*

$$N(r,0;f(z+c))\leqslant N(r,0;f(z))+S(r,f(z)),\quad N(r,\infty;f(z+c))\leqslant N(r,\infty;f(z))+S(r,f(z)),$$
$$\overline{N}(r,0;f(z+c))\leqslant \overline{N}(r,0;f(z))+S(r,f(z)),\quad \overline{N}(r,\infty;f(z+c))\leqslant \overline{N}(r,\infty;f(z))+S(r,f(z)).$$

**Lemma 2.4.** *Let $g$ be a meromorphic function of finite order $\rho$, and let $c\in\mathbb{C}^*$ be fixed. Then*

$$T(r,g(z+c))=T(r,g(z))+S(r,g).$$

**Proof.** The lemma can be proof in the line of the proof of [6, Theorem 2.1]. □

**Lemma 2.5.** *Let $f$ be a transcendental meromorphic function of finite order, then $S(r,\mathcal{L}_p(f,\Delta_c))$ can be replaced by $S(r,f)$.*

**Proof.** In view of Lemma 2.4, we have

$$T(r,\mathcal{L}_p(f,\Delta_c))\leqslant\sum_{j=1}^{p}T(r,f(z+cj))+T(r,f)+O(1)\leqslant(p+1)T(r,f)+O(1),$$

with this the lemma follows. □

**Lemma 2.6** [19]. *Let $f$ be a non-constant meromorphic function and $\mathcal{Q}(f)=\sum_{i=0}^{n}a_if^i$, where $a_i\in\mathbb{C}$ with $a_n\neq 0$. Then $T(r,\mathcal{Q}(f))=nT(r,f)+O(1)$.*

**Lemma 2.7** [15]. *If $N\left(r,0;f^{(k)}|f\neq 0\right)$ be the counting function of those zeros of $f^{(k)}$ which are not the zeros of $f$, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N\left(r,0;f^{(k)}|f\neq 0\right)\leqslant k\overline{N}(r,\infty;f)+N(r,0;f|<k)+k\overline{N}(r,0;f|\geqslant k)+S(r,f).$$

**Lemma 2.8.** *Let $\mathcal{F}$ and $\mathcal{G}$ share $(1,t)$, $1\leqslant t<\infty$ and $(\infty,0)$, then*

$$\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\leqslant\frac{1}{t+1}\left\{\overline{N}(r,0;\mathcal{F})+\overline{N}(r,0;\mathcal{G})\right\}+\frac{2}{t+1}\overline{N}(r,\infty;\mathcal{F})+S(r,f).$$

**Proof.** In view of Lemma 2.5 and 2.7, one must have

$$\begin{aligned}\overline{N}_*(r,1;\mathcal{F},\mathcal{G})&=\overline{N}_L(r,1;\mathcal{F})+\overline{N}_L(r,1;\mathcal{G})\leqslant\overline{N}(r,1;\mathcal{F}|\geqslant t+2)+\overline{N}(r,1;\mathcal{G}|\geqslant t+2)\\&\leqslant\frac{1}{t+1}\left\{N(r,0;\mathcal{F}'|\mathcal{F}\neq 0)+N(r,0;\mathcal{G}'|\mathcal{G}\neq 0)\right\}\\&\leqslant\frac{1}{t+1}\left\{\overline{N}(r,0;\mathcal{F})+\overline{N}(r,0;\mathcal{G})+2\overline{N}(r,\infty;\mathcal{F})\right\}+S(r,f).\end{aligned}$$

This completes the proof of the lemma. □

**Lemma 2.9.** Let $\mathcal{F}$ and $\mathcal{G}$ share $(1,t)$, $1\leqslant t<\infty$ and $(\infty,0)$, then

$$\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\leqslant\frac{1}{t}\left\{\overline{N}(r,0;\mathcal{F})+N(r,\infty;\mathcal{F})\right\}+S(r,\mathcal{F})+S(r,f).$$

**Proof.** In view of Lemma 2.7, we have

$$\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\leqslant\overline{N}(r,1;\mathcal{F}|\geqslant t+1)\leqslant\frac{1}{t}N(r,0;\mathcal{F}'|\mathcal{F}=1).$$

We omit the details since rest of the proof follows the line of the proof of *Lemma 2.8*. □

**Lemma 2.10.** *For a meromorphic function $f$, we suppose that $F$ and $G$ be given as in (2.1) and $\Psi\not\equiv 0$. If $f$ and $\mathcal{L}_p(f,\Delta_c)$ share $(\infty,k)$, where $0\leqslant k<\infty$ and $\mathcal{F}$, $\mathcal{G}$ share $(1,t)$, then*

$$\begin{aligned}&\left\{n(k+1)-1\right\}\overline{N}(r,\infty;f|\geqslant k+1)\\&\leqslant\frac{t+2}{t+1}\left\{\overline{N}(r,0;f)+\overline{N}(r,0;\mathcal{L}_p(f,\Delta_c))\right\}+\frac{2}{t+1}\overline{N}(r,\infty;f)+S(r,f).\end{aligned}$$

**Proof.** It is clear that $\mathcal{F}$ and $\mathcal{G}$ share $(\infty,nk)$ since $f$ and $\mathcal{L}_p(f,\Delta_c)$ share $(\infty,k)$. Let $z_0$ be a pole of $\mathcal{F}$ of multiplicity $q(\geqslant nk+1)$, then $z_0$ must be a pole of $\mathcal{G}$ of multiplicity $r(\geqslant nk+1)$ and conversely. Again one may note that there is no pole of $\mathcal{F}$ and $\mathcal{G}$ of multiplicity $q$, where $nk<q<n(k+1)$. Next by using Lemmas 2.5, 2.6 and 2.8, we get from the definition of $\Psi$ that

$$\begin{aligned}&\left\{nk+n-1\right\}\overline{N}(r,\infty;f|\geqslant k+1)\leqslant N(r,0;\Psi)\leqslant N(r,\infty;\Psi)+S(r,\mathcal{F})+S(r,\mathcal{G})\\&\leqslant\overline{N}(r,0;\mathcal{F})+\overline{N}(r,0;\mathcal{G})+\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+S(r,\mathcal{F})+S(r,\mathcal{G})\leqslant\overline{N}(r.0;f)\\&+\overline{N}(r,0;\mathcal{L}_p(f,\Delta_c))+\frac{1}{t+1}\left\{\overline{N}(r,0;f)+\overline{N}(r,0;\mathcal{L}_p(f,\Delta_c))+2\overline{N}(r,\infty;f)\right\}+S(r,f)\\&\leqslant\frac{t+2}{t+1}\left\{\overline{N}(r,0;f)+\overline{N}(r,0;\mathcal{L}_p(f,\Delta_c))\right\}+\frac{2}{t+1}\overline{N}(r,\infty;f)+S(r,f).\end{aligned}$$

This completes the proof of the lemma. □

**Lemma 2.11** [17, 20]. *If $\mathcal{F}$ and $\mathcal{G}$ share $(\infty,0)$ and $\Psi\equiv 0$, then $\mathcal{F}\equiv\mathcal{G}$.*

**Lemma 2.12** [18]. *Let $\mathcal{H}\equiv 0$ and $\mathcal{F}$, $\mathcal{G}$ share $(\infty,0)$, then $\mathcal{F}$ and $\mathcal{G}$ share $(1,\infty)$, $(\infty,\infty)$.*

**Lemma 2.13** [1]. *Let $\mathcal{F}$, $\mathcal{G}$ be two meromorphic functions sharing $(1,2)$ and $(\infty,k)$, where $0\leqslant k\leqslant\infty$. Then one of the following cases holds.*

(i) $T(r,\mathcal{F})+T(r,\mathcal{G}) \leqslant 2\{N_2(r,0;\mathcal{F})+N_2(r,0;\mathcal{G})+\overline{N}(r,\infty;\mathcal{F})+N(r,\infty;\mathcal{G})+\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\}+S(r,\mathcal{F})+S(r,\mathcal{G})$.

(ii) $\mathcal{F}\equiv\mathcal{G}$.

(iii) $\mathcal{F}\mathcal{G}\equiv 1$.

## 3. PROOFS OF THE MAIN RESULTS

*Proof. of Theorem 1.1.* Let $\mathcal{F}$ and $\mathcal{G}$ be given by (2.1).

Now we discuss the following two cases.

**Case 1.** Let us suppose that $\mathcal{H}\not\equiv 0$. Then in view of Lemma 2.11, we have $\Psi\not\equiv 0$. Since $E_f(\mathcal{S}_a^n,1)=E_{\mathcal{L}_p(f,\Delta_c)}(\mathcal{S}_a^n,1)$ and $E_f(\mathcal{S}_2,0)=E_{\mathcal{L}_p(f,\Delta_c)}(\mathcal{S}_2,0)$, it follows that $\mathcal{F}$ and $\mathcal{G}$ share $(1,1)$ and $(\infty,0)$ . By the *Second Fundamental Theorem*, we get

$$\begin{aligned}T(r,\mathcal{F})+T(r,\mathcal{G}) &\leqslant \overline{N}(r,1;\mathcal{F})+\overline{N}(r,0;\mathcal{F})+\overline{N}(r,\infty;\mathcal{F})+\overline{N}(r,1;\mathcal{G})+\overline{N}(r,0;\mathcal{G})\\ &+\overline{N}(r,\infty;\mathcal{G})-\overline{N}_0(r,0;\mathcal{F}')-\overline{N}_0(r,0;\mathcal{G}')+S(r,\mathcal{F})+S(r,\mathcal{G}).\end{aligned}$$

Using Lemma 2.6 and Lemmas 2.1, 2.2, 2.3 of [2, p. 384], we get

$$\begin{aligned}n\Big\{T(r,f)+T(r,\mathcal{L}_p(f,\Delta_c))\Big\} &\leqslant 4\Big\{\overline{N}(r,0;f)+\overline{N}(r,0;\mathcal{L}_p(f,\Delta_c))\Big\}\\ &+6\overline{N}(r,\infty;f)-2\left(t-\frac{3}{2}\right)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+S(r,\mathcal{F})+S(r,\mathcal{G}). \end{aligned} \tag{3.1}$$

Applying Lemma 2.8 with $t=1$ and Lemma 2.10 with $t=1$, $k=0$, we get from (3.1) that

$$\begin{aligned}&n\Big\{T(r,f)+T(r,\mathcal{L}_p(f,\Delta_c))\Big\}\\ &\leqslant \frac{9}{2}\Big\{\overline{N}(r,0;f)+\overline{N}(r,0;\mathcal{L}_p(f,\Delta_c))\Big\}+7\overline{N}(r,\infty;f)+S(r,f)+S(r,\mathcal{L}_p(f,\Delta_c))\\ &\leqslant \left(\frac{9}{2}+\frac{21}{2(n-2)}\right)\Big\{\overline{N}(r,0;f)+\overline{N}(r,0;\mathcal{L}_p(f,\Delta_c))\Big\}+S(r,f)+S(r,\mathcal{L}_p(f,\Delta_c))\\ &\leqslant \left(\frac{9}{2}+\frac{21}{2(n-2)}\right)\Big\{T(r,f)+T(r,\mathcal{L}_p(f,\Delta_c))\Big\}+S(r,f)+S(r,\mathcal{L}_p(f,\Delta_c)).\end{aligned} \tag{3.2}$$

which contradicts $n\geqslant 7$.

**Case 2.** Let us suppose that $\mathcal{H}\equiv 0$. On integration twice, we get

$$\mathcal{F}=\frac{\mathcal{A}\mathcal{G}+\mathcal{B}}{\mathcal{C}\mathcal{G}+\mathcal{D}}, \tag{3.3}$$

where $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\in\mathbb{C}$ such that $\mathcal{A}\mathcal{D}-\mathcal{B}\mathcal{C}\neq 0$.

We now discuss the following two cases.

*Case 2a.* Let $\mathcal{A}\mathcal{C}\neq 0$. We thus see that $\mathcal{A}\neq 0$ and $\mathcal{C}\neq 0$.

It follows from (3.3) that

$$\mathcal{F}-\frac{\mathcal{A}}{\mathcal{C}}=\frac{\mathcal{B}\mathcal{C}-\mathcal{A}\mathcal{D}}{\mathcal{C}(\mathcal{C}\mathcal{G}+\mathcal{D})}. \tag{3.4}$$

Clearly it follows from (3.4) that all the zeros of $\mathcal{F}-\dfrac{\mathcal{A}}{\mathcal{C}}$ corresponds from the poles of $\mathcal{G}$. We also see from our hypothesis that $\mathcal{F}$ and $\mathcal{G}$ share $(\infty,\infty)$, so from (3.3) we see that $\infty$ is an e.v.P of $\mathcal{G}$. In other words $\mathcal{F}$ omits the value $\frac{\mathcal{A}}{\mathcal{C}}$.

By the *Second Fundamental Theorem*, we get

$$nT(r,f)\leqslant \overline{N}(r,0;\mathcal{F})+\overline{N}(r,\infty;\mathcal{F})+\overline{N}\left(r,\frac{\mathcal{A}}{\mathcal{C}};\mathcal{F}\right)+S(r,\mathcal{F})$$

$$= \overline{N}(r,0;f) + S(r,f) \leqslant T(r,f) + S(r,f),$$

which contradicts $n \geqslant 7$.

*Case 2b.* Let $\mathcal{AC} = 0$. This shows that one of $\mathcal{A}$ and $\mathcal{C}$ is zero, otherwise for $\mathcal{A} = 0 = \mathcal{C}$ leads the function $\mathcal{F}$ to be a constant and which would be a contradiction.

*Subcase 2b.1.* Let $\mathcal{A} \neq 0$ and $\mathcal{C} = 0$. Then,

$$\mathcal{F} = \alpha\mathcal{G} + \beta, \tag{3.5}$$

where $\alpha = \dfrac{\mathcal{A}}{\mathcal{D}}$ and $\beta = \dfrac{\mathcal{B}}{\mathcal{D}}$.

If $\mathcal{F}$ has no 1-points, then by *Second Fundamental Theorem*, we get

$$nT(r,f) \leqslant \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,\infty;\mathcal{F}) + S(r,\mathcal{F}) \leqslant 2T(r,f) + S(r,f),$$

which contradicts $n \geqslant 7$.

If $\mathcal{F}$ and $\mathcal{G}$ both have some 1-points, then we have $\alpha + \beta = 1$.

If $\beta = 0$, then $\alpha = 1$. So we have $\mathcal{F} \equiv \mathcal{G}$. Thus we have $\mathcal{L}_p\left(f, \Delta_c\right) \equiv tf$, where $t^n = 1$ with $t \neq -1$.

Next, we suppose that $\beta \neq 0$. So it is clear that $\mathcal{F} - \beta = \alpha\mathcal{G}$. By *Second Fundamental Theorem*, we get

$$\begin{aligned} nT(r,f) &\leqslant \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,\beta;\mathcal{F}) + S(r,\mathcal{F}) \\ &\leqslant \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;\mathcal{G}) \leqslant (p+3)T(r,f) + S(r,f), \end{aligned}$$

which contradicts $n \geqslant p+4$.

*Subcase 2b.2.* Let $\mathcal{A} = 0$ but $\mathcal{C} \neq 0$. Then we have

$$\mathcal{F} = \frac{1}{\gamma\mathcal{G} + \delta}, \tag{3.6}$$

where $\gamma = \dfrac{\mathcal{C}}{\mathcal{B}}$ and $\delta = \dfrac{\mathcal{D}}{\mathcal{B}}$.

If $\mathcal{F}$ has no 1-points, then proceeding exactly same way as done in *Subcase b.1*, we arrive at a contradiction.

If $\mathcal{F}$ and $\mathbb{G}$ have some 1-points, then it follows from (3.6) that $\gamma + \delta = 1$.

We now see from (3.6) that

$$\mathcal{F} = \frac{1}{\gamma\mathcal{G} + 1 - \gamma}. \tag{3.7}$$

We note that as $\mathcal{C} \neq 0$, $\gamma \neq 0$. Suppose $\delta \neq 0$. So $\gamma \neq 1$. Since $\mathcal{F}$ and $\mathcal{G}$ share $(\infty,\infty)$, so from (3.7), we see that $\mathcal{F}$ and $\mathcal{G}$ omit $\infty$.

By the *Second Fundamental Theorem*, we get

$$\begin{aligned} nT(r,f) = T(r,\mathcal{F}) &\leqslant \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,\infty;\mathcal{F}) + \overline{N}\left(r, \frac{1}{1-\gamma}; \mathcal{F}\right) + S(r,\mathcal{F}) \\ &\leqslant \overline{N}(r,0;f) + \overline{N}(r,0;\mathcal{G}) + S(r,f) \leqslant (p+2)T(r,f) + S(r,f), \end{aligned}$$

which contradicts $n \geqslant p+4$.

Next we suppose that $\delta = 0$. Therefore $\gamma = 1$. Then we get $\mathcal{FG} \equiv 1$, i.e., $f\left(\mathcal{L}_p(f,\Delta_c)\right) \equiv \theta a^2$, where $\theta^n = 1$.

Next since $\mathcal{F}$ and $\mathcal{G}$ share $(\infty,\infty)$, so we have $N\left(r, \dfrac{\mathcal{L}_p(f,\Delta_c)}{f}\right) = N(r,0;f)$ and so in view of Lemma 2.1, we get

$$\begin{aligned} 2T(r,f) &\leqslant T\left(r, \frac{\theta a^2}{f^2}\right) + S(r,f) \leqslant T\left(r, \frac{\mathcal{L}_p(f,\Delta_c)}{f}\right) + S(r,f) \\ &\leqslant N\left(r, \frac{\mathcal{L}_p(f,\Delta_c)}{f}\right) + S(r,f) \leqslant N(r,0;f) + S(r,f) \leqslant T(r,f) + S(r,f), \end{aligned}$$

which is a contradiction.

This completes the proof of *Theorem 1.1*. □

**Proof. of Theorem 1.2.** Let $\mathcal{F}$ and $\mathcal{G}$ be given by (2.1) and $\Psi \not\equiv 0$, since otherwise the proof follows from the Lemma 2.11. Again Since $E_f(\mathcal{S}_a^n, 2) = E_{\mathcal{L}_p(f,\Delta_c)}(\mathcal{S}_a^n, 2)$ and $E_f(\mathcal{S}_2, 0) = E_{\mathcal{L}_p(f,\Delta_c)}(\mathcal{S}_2, 0)$, so it follows that $\mathcal{F}$, $\mathcal{G}$ share $(1, 2)$ and $(\infty, 0)$. Let if possible (i) of Lemma 2.13 holds. Then with the help of Lemma 2.6, one must have

$$\begin{aligned} & n\Big\{T(r,f) + T(r,\mathcal{L}_p(f,\Delta_c))\Big\} \\ \leqslant\ & 4\Big\{\overline{N}(r,0;f) + \overline{N}(r,0;\mathcal{L}_p(f,\Delta_c))\Big\} + 6\overline{N}(r,\infty;f) + S(r,\mathcal{F}) + S(r,\mathcal{G}). \end{aligned} \tag{3.8}$$

Now with the help of Lemma 2.10 with $t = 2$, $k = 0$, we get from (3.8)

$$\begin{aligned} & n\Big\{T(r,f) + T(r,\mathcal{L}_p(f,\Delta_c))\Big\} \\ \leqslant\ & \left(4 + \frac{24}{3n-5}\right)\Big\{T(r,f) + T(r,\mathcal{L}_p(f,\Delta_c))\Big\} + S(r,f) + S(r,\mathcal{L}_p(f,\Delta_c)), \end{aligned}$$

which contradicts $n \geqslant 6$.

Now the rest of the proof follows from the line of the proof of Theorem 1.1. □

## 4. PROOFS OF THE COROLLARIES

**Proof of Corollary 1.1.** Let us suppose that $\mathcal{F} = f^n$ and $\mathcal{G} = \left(\overset{s}{\Delta}_c\right)^n$. Then, following the same procedure as adopted in the proof of *Theorem 1.1*, we obtain

$$\overset{s}{\Delta}_c \equiv tf. \tag{4.1}$$

**Proof of Corollary 1.2.** The proof can be carried out exactly the line of the proof of *Theorem 1.2* and that of *Corollary 1.1*.

**Proof of Remark 1.4.** Since the distinct roots of $\displaystyle\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}z^j = t$ are $\alpha_j = 1 + |t|^{\frac{1}{s}} e^{\frac{\theta+2\pi ij}{s}}$, where $-\pi < \theta \leqslant \pi$, $j = 0, 1, \ldots, s-1$, therefore the general solution of the relation $\overset{s}{\Delta}_c f \equiv tf$ will be of the form

$$f(z) = \pi_{s-1}(z)\alpha_{s-1}^{\frac{z}{c}} + \ldots + \pi_0(z)\alpha_0^{\frac{z}{c}}.$$

**Verification**:

$$\begin{aligned} \overset{s}{\Delta}_c f &= \binom{s}{0}f(z+sc) - \binom{s}{1}f(z+(s-1)c) + \ldots + (-1)^s\binom{s}{s}f(z) \\ &= \binom{s}{0}\Big\{\pi_{s-1}(z+sc)\alpha_{s-1}^{\frac{z}{c}}\alpha_{s-1}^{p} + \ldots + \pi_0(z+sc)\alpha_0^{\frac{z}{c}}\alpha_0^{s}\Big\} \\ &- \binom{s}{1}\Big\{\pi_{s-1}(z+(s-1)c)\alpha_{s-1}^{\frac{z}{c}}\alpha_{s-1}^{s-1} + \ldots + \pi_0(z+(s-1)c)\alpha_0^{\frac{z}{c}}\alpha_0^{s-1}\Big\} \\ &+ \ldots + \binom{s}{s}(-1)^s\Big\{\pi_{s-1}(z)\alpha_{s-1}^{\frac{z}{c}} + \ldots + \pi_0(z)\alpha_0^{\frac{z}{c}}\Big\} \\ &= \Big\{\binom{s}{0}\alpha_{s-1}^{s} - \binom{s}{1}\alpha_{s-1}^{s-1} + \ldots + \binom{s}{s}(-1)^s\Big\}\pi_{s-1}(z)\alpha_{s-1}^{\frac{z}{c}} \\ &+ \ldots + \Big\{\binom{s}{0}\alpha_0^{s} - \binom{s}{1}\alpha_0^{s-1} + \ldots + \binom{s}{s}(-1)^s\Big\}\pi_0(z)\alpha_0^{\frac{z}{c}} \\ &= (\alpha_{s-1}-1)^s\,\pi_{s-1}(z)\alpha_{s-1}^{\frac{z}{c}} + \ldots + (\alpha_0-1)^s\,\pi_0(z)\alpha_0^{\frac{z}{c}} \end{aligned}$$

$$= \left(|t|^{\frac{1}{s}} e^{\frac{\theta+2(s-1)\pi i}{s}}\right)^s \pi_{s-1}(z)\alpha_{s-1}^{\frac{z}{c}} + \ldots + \left(|t|^{\frac{1}{s}} e^{\frac{\theta+oi}{s}}\right)^s \pi_0(z)\alpha_0^{\frac{z}{c}}$$

$$= t\left\{\pi_{s-1}(z)\alpha_{s-1}^{\frac{z}{c}} + \ldots + \pi_0(z)\alpha_0^{\frac{z}{c}}\right\} = tf(z).$$

## 5. CONCLUDING REMARKS

In this section, we have the following observation.

**Observation 5.1.** A non-constant finite ordered meromorphic function satisfying the relation

$$\mathcal{L}_p(f, \Delta_c) \equiv tf \tag{5.1}$$

must assume the following form

$$f(z) = \pi_p(z)\alpha_p^{\frac{z}{c}} + \ldots + \pi_1(z)\alpha_1^{\frac{z}{c}},$$

where $\pi_j(z), (j = 1, \ldots, p) \in \mathbb{P}_c$, and $\alpha_j (j = 1, \ldots, p)$ are the roots of the equation

$$a_p w^p + a_{p-1} w^{p-1} + \ldots + a_1 w - \left(\sum_{j=1}^{p} a_j + t\right) = 0.$$

For $p = 1$, we have $\mathcal{L}_1(f, \Delta_c) \equiv tf$, which implies that $f(z+c) = \left(\frac{a_1 + t}{a_1}\right) f(z)$.

Clearly, in this case the general solution of (5.1) is

$$f(z) = \pi_1(z)\left(\frac{a_1 + t}{a_1}\right)^{\frac{z}{c}} = \pi_1(z)\alpha_1^{\frac{z}{c}},$$

where $\alpha_1$ is a root of the equation $a_1 w - (a_1 + t) = 0$.

**Verification**:

$$\mathcal{L}_1(f, \Delta_c) = a_1 f(z+c) - (a_1) f(z) = a_1\left\{\pi_1(z+c)\alpha_1\alpha_1^{\frac{z}{c}}\right\} - a_1\left\{\pi_1(z)\alpha_1^{\frac{z}{c}}\right\}$$

$$= \left\{a_1\alpha_1 - a_1\right\}\pi_1(z)\alpha_1^{\frac{z}{c}} = t\pi_1(z)\alpha_1^{\frac{z}{c}} = tf(z).$$

For $p = 2$, we have $\mathcal{L}_2(f, \Delta_c) \equiv tf$, which in turn implies that

$$a_2 f(z+2c) + a_1 f(z+c) - (a_1 + a_2 + t) f(z) \equiv 0.$$

Let $\alpha_1, \alpha_2$ be the roots of the equation

$$a_2 w^2 + a_1 w - (a_1 + a_2 + t) = 0.$$

Then

$$\alpha_1, \alpha_2 = \frac{-a_1 \pm \sqrt{a_1^2 + 4a_2(a_1 + a_2 + t)}}{2a_2}.$$

In this case the general solution of (5.1) is

$$f(z) = \pi_1(z)\left(\frac{-a_1 + \sqrt{a_1^2 + 4a_2(a_1 + a_2 + t)}}{2a_2}\right)^{\frac{z}{c}}$$

$$+ \pi_2(z)\left(\frac{-a_1 - \sqrt{a_1^2 + 4a_2(a_1 + a_2 + t)}}{2a_2}\right)^{\frac{z}{c}} = \pi_1(z)\alpha_1^{\frac{z}{c}} + \pi_2(z)\alpha_2^{\frac{z}{c}}.$$

Lets verify the above fact.

$$\mathcal{L}_2(f,\Delta_c) = a_2 f(z+2c) + a_1 f(z+c) - (a_1+a_2)f(z) = a_2\left\{\pi_1(z+2c)\lambda_1^2\lambda_1^{\frac{z}{c}} + \pi_2(z+2c)\lambda_2^2\lambda_2^{\frac{z}{c}}\right\}$$

$$+ a_1\left\{\pi_1(z+c)\lambda_1\lambda_1^{\frac{z}{c}} + \pi_2(z+c)\lambda_2\lambda_2^{\frac{z}{c}}\right\} - (a_1+a_2)\left\{\pi_1(z)\lambda_1^{\frac{z}{c}} + \pi_2(z)\lambda_2^{\frac{z}{c}}\right\}$$

$$= \left\{a_2\lambda_1^2 + a_1\lambda_1 - (a_1+a_2)\right\}\pi_1(z)\lambda_1^{\frac{z}{c}} + \left\{a_2\lambda_2^2 + a_1\lambda_2 - (a_1+a_2)\right\}\pi_2(z)\lambda_2^{\frac{z}{c}}$$

$$= t\pi_1(z)\lambda_1^{\frac{z}{c}} + t\pi_2(z)\lambda_2^{\frac{z}{c}} = t\left\{\pi_1(z)\lambda_1^{\frac{z}{c}} + \pi_2(z)\lambda_2^{\frac{z}{c}}\right\} = tf(z).$$

So we conjecture that the general solution of the relation (5.1) is

$$f(z) = \pi_p(z)\alpha_p^{\frac{z}{c}} + \pi_{p-1}(z)\alpha_{p-1}^{\frac{z}{c}} + \ldots + \pi_1(z)\alpha_1^{\frac{z}{c}},$$

where $\pi_j(z)(j=1,\ldots,p) \in \mathbb{P}_c$, and $\alpha_j(j=1,\ldots,p)$ are the roots of the equation

$$a_p w^p + a_{p-1}w^{p-1} + \ldots + a_1 w - \left(\sum_{j=1}^{p} a_j + t\right) = 0.$$

But unfortunately we have not succeeded to prove it.

**An open question.** What would be the general meromorphic solution of the difference equation $\mathcal{L}_p(f,\Delta) \equiv tf$?

## REFERENCES


1. A. Banerjee, "Meromorphic functions sharing two sets," Czech. Math. J. **57** (132), 1199–1214 (2007).
2. A. Banerjee, "Uniqueness of meromorphic functions sharing two sets with finite weight II," Tamkang J. Math. **41** (4), 379–392 (2010).
3. A. Banerjee and M. B. Ahamed, "On uniqueness of meromorphic functions sharing three sets with finite weights," Bull. Polish Acad. Sci. Math. **62** (3), 243–256 (2014).
4. A. Banerjee and S. Bhattacharyya, "On the uniqueness of meromorphic functions and its difference operator sharing values or sets," Rend. Circ. Mat. Palermo, II. https://doi.org/10.1007/s12215-016-0295-1.
5. B. Chen and Z. Chen, "Meromorphic functions sharing two sets with its difference operator," Bull. Malays. Math. Soc. **35** (3), 765–774 (2012).
6. Y. M. Chiang and S. J. Feng, "On the Nevanlinna characteristic $f(z+Q)$ and difference equation in complex plane," Ramanujan J. **16**, 105–129 (2008).
7. B. Deng, D. Liu and D. Yang, "Meromorphic functions and its difference operator share two sets with weight k," Turkish J. Math. **41**, 1155–1163 (2017).
8. R. G. Halburd and R. J. Korhonen, "Nevanlinna theory for the difference operator," Ann. Acad. Sci. Fenn. Math. **31** (2), 463–478 (2006).
9. W. K. Hayman, *Meromorphic Functions* ( Clarendon Press, Oxford, 1964).
10. J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo and J. L. Zhang, "Value sharing results for shifts of meromorphic function and sufficient conditions for periodicity," J. Math. Anal. Appl. **355**, 352–363 (2009).
11. I. Lahiri, "Weighted sharing and uniqueness of meromorphic functions," Nagoya Math. J. **161**, 193–206 (2001).
12. I. Lahiri, "Weighted value sharing and uniqueness of meromorphic functions," Complex Var. Theory Appl. **46**, 241–253 (2001).
13. I. Lahiri, "Value distribution of certain differential polynomials," Int. J. Math. Math. Sci. **28** (2), 83–91 (2001).
14. I. Lahiri and A. Banerjee, "Weighted sharing of two sets," Kyungpook Math. J. **46** (1), 79–87 (2006).
15. I. Lahiri and S. Dewan, "Value distribution of the product of a meromorphic function and its derivative," Kodai Math. J. **26**, 95–100 (2003).
16. I. Laine, *Nevanlinna Theory and Complex Differential Equations* (Walter de Gruyter, Berlin, 1993).
17. H. X. Yi, "Meromorphic functions that share three sets," Kodai Math. J. **20**, 22–32 (1997).
18. H. X. Yi, "Meromorphic functions that share one or two values II," Kodai Math. J. **22**, 264–272 (1999).
19. C. C. Yang, "On deficiencies of differential polynomials II," Math. Z. **125**, 107–112 (1972).
20. H. X. Yi and L. Z. Yang, "Meromorphic functions that share two sets," Kodai Math. J. **20**, 127–134 (1997).
21. J. Zhang, "Value distribution and shared sets of differences of meromorphic functions," J. Math. Anal. Appl. **367** (2), 401–408 (2010).