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UNIQUENESS OF MEROMORPHIC FUNCTION WITH ITS SHIFT OPERATOR UNDER THE PURVIEW OF TWO OR THREE SHARED SETS

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ABSTRACT. Taking two and three shared set problems into background, the uniqueness problem of a meromorphic function together with its shift operator have been studied. Our results will improve a number of recent results in the literature. Some examples have been provided in the last section to show that certain conditions used in the paper, is the best possible.

1. Introduction, definitions and results

In the paper we will denote by $\mathbb{C}$ the set of all complex numbers, by $\mathbb{N}$ the set of all positive integers and by $\overline{\mathbb{C}} =: \mathbb{C} \cup \infty$. Also it is assumed without stating it explicitly that all considered meromorphic functions are defined on $\mathbb{C}$ and that they are non-constant.

For such a function f and $a \in \overline{\mathbb{C}}$, each z with $f(z) = a$ will be called a-point of f. For a meromorphic function f and a set $S \subset \overline{\mathbb{C}}$ we define $E_f(S)$ ($\overline{E}_f(S)$) as the set of all a-points of f, when $a \in S$, together with their multiplicity (without their multiplicity). If $E_f(S) = E_g(S)$ ($\overline{E}_f(S) = \overline{E}_g(S)$) then we simply say f, g share S Counting Multiplicities or CM (Ignoring Multiplicities or IM).

Lahiri [10, 11] introduced the following notion of weighted sharing of values and sets. It has become an useful tool to find new directions of research in the uniqueness theory.

Definition 1.1 ([10,11])**.** Let k be a non-negative integer or infinity. For $a \in \mathbb{C}\cup\{\infty\}$, we denote by $E_f(a,k)$, the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \le k$ and $k+1$ times if $m > k$. If $E_f(a,k) = E_g(a,k)$, we see that f and g share the value a with weight k.

We write f and g share (a,k) to mean that f and g share the value a with weight k.

Definition 1.2 ([10,11])**.** Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and k be a non-negative integer or ∞. We denote by $E_f(S,k)$ the set $\bigcup\limits_{a\in S} E_f(a,k)$. If $\bigcup\limits_{a\in S} E_f(a,k) = \bigcup\limits_{a\in S} E_g(a,k)$, then we say that f and g share the set S with k.

It was Fujimoto [7], who first discovered a special property of a polynomial, reasonably called as critical injection property though initially Fujimoto [7] called it as property (H). Critical injection property of a polynomial may be stated as follows: A Polynomial P is said to satisfy property if $P(\alpha) \ne P(\beta)$ for any two distinct zeros of α and β of the derivative P'.

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Clearly, the meaning of critical injection property is that the polynomial P is injective on the set of distinct zeros of P'. Naturally a polynomial with property may be called a critical injective polynomial.

For a non-constant meromorphic function, we define its shift and difference operator respectively by $f(z+\omega)$ and $\Delta_\omega f = f(z+\omega) - f(z)$, where ω is a non-zero constant.

Recently among the researchers [4–6, 13, 15], an increasing interest has been found to find the possible relationship between a meromorphic function $f(z)$ and its shift $f(z+\omega)$ or its difference $\Delta_c f$.

However in [5, 15], the authors were unable to obtain the uniqueness relationship of a meromorphic function with its shift operator for the specific choice of the sets.

So quest for those range sets, shared by a function and its shift operator, for which they become identical, is a natural phenomenon. Earlier several authors considered uniqueness problem between two meromorphic functions f and g sharing two sets. But in this particular direction, the first investigation for uniqueness of a meromorphic function and its shift was due to Zhang [15].

In 2010, Zhang [15] proved the following theorem.

Theorem A ([15]). *Let $m \geq 2$, $n \geq 2m+4$ with n and $n-m$ having no common factors. Let a and b be two non-zero constant such that the equation $w^n + aw^{n-m} + b = 0$ has no multiple roots. Let $S = \{w : w^n + aw^{n-m} + b = 0\}$. Suppose that $f(z)$ is a non-constant meromorphic function of finite order. Then $E_{f(z)}(S,\infty) = E_{f(z+\omega)}(S,\infty)$ and $E_{f(z)}(\{\infty\},\infty) = E_{f(z+\omega)}(\{\infty\},\infty)$ imply that $f(z) \equiv f(z+\omega)$.*

Qi-Dou-Yang [13] studied the case $m = 1$ in the above theorem and with the aid of some extra supposition reduced the lower bound of the range set as follows.

Theorem B ([13]). *Let $n \geq 6$ be an integer and let a, b be two non-zero constants such that the equation $w^n + aw^{n-1} + b = 0$ has no multiple roots. Denote $S = \{w : w^n + aw^{n-1} + b = 0\}$. Suppose f is a non-constant meromorphic function of finite order. Then $E_{f(z)}(S,\infty) = E_{f(z+\omega)}(S,\infty)$, $E_{f(z)}(\{\infty\},\infty) = E_{f(z+\omega)}(\{\infty\},\infty)$ and $\overline{N}(r,f) \leq \dfrac{n-3}{n-1}T(r,f) + S(r,f)$ implies that $f(z) \equiv f(z+\omega)$.*

As in Theorem A, $\gcd(n,m) = 1$, so we see that the lower bound of cardinality of the range set considered in Theorem A, is 9, and that in Theorem B, is 6. However, in 2013, Bhoosnurmath-Kabbur [4] improved Theorem A by reducing the lower bound of the cardinality of range set and obtained the following result.

Theorem C ([4]). *Let $n \geq 8$ be an integer and $c(\neq 0,\ 1)$ is a constant such that the equation $P(w) = \dfrac{(n-1)(n-2)}{2}w^n - n(n-2)w^{n-1} + \dfrac{n(n-1)}{2}z^{n-2} - c$. Let us suppose that $S = \{w : P(w) = 0\}$ and f is a non-constant meromorphic functions of finite order, then $E_{f(z)}(S,\infty) = E_{f(z+\omega)}(S,\infty)$ and $E_{f(z)}(\{\infty\},\infty) = E_{f(z+\omega)}(\{\infty\},\infty)$ imply that $f(z) \equiv f(z+\omega)$.*

By considering "entire" function, Bhoosnurmath-Kabbur [4] obtained the following result.

Theorem D ([4]). *Let $n \geq 7$ be an integer and $c(\neq 0,\ 1)$ is a constant such that the equation $P(w) = \dfrac{(n-1)(n-2)}{2}w^n - n(n-2)w^{n-1} + \dfrac{n(n-1)}{2}z^{n-2} - c$. Let us suppose that $S = \{w : P(w) = 0\}$ and f is a non-constant entire functions of finite order, then $E_{f(z)}(S,\infty) = E_{f(z+\omega)}(S,\infty)$ and $E_{f(z)}(\{\infty\},\infty) = E_{f(z+\omega)}(\{\infty\},\infty)$ imply that $f(z) \equiv f(z+\omega)$.*

Next we point out that in the above mentioned paper, the lower bound of cardinality of the main range set for meromorphic function has always been stick to 8 without the help of any extra conditions. Also we note that in none of the papers the authors were engaged to relax the nature of sharing the range set.

Considering all the above facts, the following question may appear in one's mind.

Question 1.1. For the case of two sets sharing, can the lower bound of the range set be further diminish?

Question 1.2. Is it possible to relax also the nature of sharing of the range sets?

The purpose of the paper is to deal with the above two questions.

In fact, in the paper we shall show that the lower bound of the main range set can significantly be reduced at the expense of replacing the second set namely the set of poles by a new one.

We would also like to investigate the situation of further diminishing the cardinality of the main range set at the cost of considering three shared sets problems.

Next for all $n \in \mathbb{N}, a, b, c \in \mathbb{C}$ we define

$$\mathcal{Q}(z) =: az^2 + bz + c, \qquad c = \frac{\{b(n-1)\}^2}{4an(n-2)}; \qquad \mathcal{P}(z) =: z^{n-2}\mathcal{Q}(z), \qquad \delta^n_{a,b} = \frac{b(1-n)}{2na}.$$

We also let

$$\sigma = -a\left(\delta^n_{a,b}\right)^{n-2}(\delta^n_{a,b} - \alpha)(\delta^n_{a,b} - \beta),$$

where α and β be the distinct roots of the equation $az^2 + bz + c = 0$.

For fixed $n \geq 3$ we will also denote by d a complex number such that $d \in \mathbb{C} \setminus \left\{0, \frac{\sigma}{2}, \sigma\right\}$.

Since $\frac{b^2}{4ac} = \frac{n(n-2)}{(n-1)^2}$, we get

$$\begin{aligned}\mathcal{P}'(z) &= naz^{n-1} + b(n-1)z^{n-2} + c(n-2)z^{n-3} = z^{n-3}\left\{naz^2 + b(n-1)z + c(n-2)\right\}\\ &= naz^{n-3}\left\{z^2 + \frac{b(n-1)}{na}z + \frac{b^2(n-1)^2}{4a^2n^2}\right\} = naz^{n-3}\left(z + \frac{b(n-1)}{2na}\right)^2\\ &= naz^{n-3}\left(z - \delta^n_{a,b}\right)^2.\end{aligned}$$

We are now at a stage to state the main result of the paper as follows.

Theorem 1.1. *Let* $S_1 = \left\{0, \delta^n_{a,b}\right\}$, $S_2 = \{z : \mathcal{P}(z) + d = 0\}$, *where* $n \geq 5$, a, b, $c \in \mathbb{C}$, $\frac{b^2}{4ac} = \frac{n(n-2)}{(n-1)^2}$, $d \in \mathbb{C} \setminus \left\{0, \frac{\sigma}{2}, \sigma\right\}$. *Let* $f(z)$ *be a finite order meromorphic function satisfying*

(i) $E_{f(z)}(S_1, 1) = E_{f(z+\omega)}(S_1, 1)$ *and* $E_{f(z)}(S_2, 3) = E_{f(z+\omega)}(S_2, 3)$, *or*

(ii) $E_{f(z)}(S_1, 2) = E_{f(z+\omega)}(S_1, 2)$ *and* $E_{f(z)}(S_2, 2) = E_{f(z+\omega)}(S_2, 2)$,

then $f(z) \equiv f(z+\omega)$.

Noting that $\frac{b^2}{4ac} = \frac{n(n-2)}{(n-1)^2} \neq 1$, i.e., $b^2 - 4ac \neq 0$, we have the following corollary.

Corollary 1.1. *Let* $S_1 = \left\{0, -\frac{2b}{5a}\right\}$ *and* $S_2 = \left\{z : az^5 + bz^4 + \frac{4b^2}{15a}z^3 + d = 0\right\}$, *where* $d \in \mathbb{C} \setminus \left\{0, \frac{8b^5}{9375a^4}, \frac{16b^5}{9375a^4}\right\}$, a, $b, \in \mathbb{C}$. *Let* $f(z)$ *be a finite order meromorphic function satisfying*

(i) $E_{f(z)}(S_1, 1) = E_{f(z+\omega)}(S_1, 1)$ *and* $E_{f(z)}(S_2, 3) = E_{f(z+\omega)}(S_2, 3)$, *or*

(ii) $E_{f(z)}(S_1, 2) = E_{f(z+\omega)}(S_1, 2)$ *and* $E_{f(z)}(S_2, 2) = E_{f(z+\omega)}(S_2, 2)$,

then $f(z) \equiv f(z+\omega)$.

Remark 1.1. If we consider "entire" function in Theorem 1.1, then the same conclusion holds for the cardinality 4 of the main range set.

Next we would like to explore the situation where the cardinality of the main range set for the case of "non-entire meromorphic" functions can further be diminished.

In this context, we have the following result.

Theorem 1.2. *Let* $S_1 = \left\{0, \delta_{a,b}^n\right\}$, $S_2 = \{z : \mathcal{P}(z) + d = 0\}$, *where* $n \geq 4$, $S_3 = \{\infty\}$, $a,\ b,\ c \in \mathbb{C}$, $\frac{b^2}{4ac} = \frac{n(n-2)}{(n-1)^2}$, $d \in \mathbb{C} \setminus \left\{0,\ \frac{\sigma}{2}, \sigma\right\}$. *If for a finite order meromorphic function* $f(z)$, $E_{f(z)}(S_1, 0) = E_{f(z+\omega)}(S_1, 0)$, $E_{f(z)}(S_2, 3) = E_{f(z+\omega)}(S_2, 3)$ *and* $E_{f(z)}(S_3, 2) = E_{f(z+\omega)}(S_3, 2)$, *then* $f(z) \equiv f(z+\omega)$.

Corollary 1.2. *Let* $S_1 = \left\{0, -\frac{3b}{8a}\right\}$ *and* $S_2 = \left\{z : az^4 + bz^3 + \frac{9b^2}{32a}z^3 + d = 0\right\}$, *where* $d \in \mathbb{C} \setminus \left\{0,\ -\frac{27b^4}{4096a^3}, -\frac{27b^4}{8192a^3}\right\}$, $a,\ b, \in \mathbb{C}$ *and* $S_3 = \{\infty\}$. *If for a finite order meromorphic function* $f(z)$, $E_{f(z)}(S_1, 0) = E_{f(z+\omega)}(S_1, 0)$, $E_{f(z)}(S_2, 3) = E_{f(z+\omega)}(S_2, 3)$ *and* $E_{f(z)}(S_3, 2) = E_{f(z+\omega)}(S_3, 2)$, *then* $f(z) \equiv f(z+\omega)$.

2. Auxiliary definitions and lemmas

The proofs of the main results depend heavily on the value distribution of meromorphic functions, which is available in [8]. We will use standard definitions and notations from this theory. In particular $N(r, a; f)$ ($\overline{N}(r, a; f)$) denotes the counting function (reduced counting function) of a-points of meromorphic functions f, $T(r, f)$ is the Nevanlinna characteristic function of f and $S(r, f)$ is used to denote each functions which is of smaller order than $T(r, f)$ when $r \to \infty$. Besides we will need the following notations.

Definition 2.1 ([9])**.** For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid= 1)$ the counting function of simple a points of f. For a positive integer m we denote by $N(r, a; f \mid\geq m))$ the counting function of those a points of f whose multiplicities are not less than m, where each a point is counted according to its multiplicity. We denote by $\overline{N}(r, a; f \mid\geq m)$, the reduced form of $N(r, a; f \mid\geq m)$.

Definition 2.2 ([14])**.** Let f and g be meromorphic functions sharing $(a, 0)$ where $a \in \mathbb{C} \cup \{\infty\}$. We denote by $\overline{N}_L(r, a; f)$ ($\overline{N}_L(r, a; g)$) the reduced counting function of those a-points of f whose multiplicity corresponding to f is bigger than that corresponding to g.

Definition 2.3 ([10,11])**.** Let f and g be non-constant meromorphic functions sharing $(a, 0)$. We denote by $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

Given meromorphic functions $f(z)$ and $f(z+\omega)$ we associate $\mathcal{F}$, $\mathcal{G}$ by

$$\mathcal{F} = \frac{\mathcal{P}(f)}{-d}, \qquad \mathcal{G} = \frac{\mathcal{P}(f(z+\omega))}{-d} \tag{2.1}$$

and to $\mathcal{F}$, $\mathcal{G}$ we associate $\mathcal{H}$ by the following formula

$$\mathcal{H} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F}-1}\right) - \left(\frac{\mathcal{G}''}{\mathcal{G}'} - \frac{2\mathcal{G}'}{\mathcal{G}-1}\right), \tag{2.2}$$

$$\Psi = \frac{\mathcal{F}'}{\mathcal{F}-1} - \frac{\mathcal{G}'}{\mathcal{G}-1}. \tag{2.3}$$

Lemma 2.1 ([11: Lemma 1])**.** *Let* $\mathcal{F}$, $\mathcal{G}$ *be meromorphic functions sharing* $(1,1)$ *and* $\mathcal{H}$ *is given by* (2.2). *If* $\mathcal{H} \not\equiv 0$, *then*

$$N(r, 1; \mathcal{F} \mid= 1) = N(r, 1; \mathcal{G} \mid= 1) \leq N(r, \mathcal{H}) + S(r, \mathcal{F}) + S(r, \mathcal{G}).$$

Next we define $\chi_n = 0$, for $n = 5$ and $\chi_n = 1$, for $n \neq 5$.

Lemma 2.2. *Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are as in (2.1), (2.2) and let S_i $i = 1, 2$ be defined as in* Theorem 1.1. *If $\mathcal{H} \not\equiv 0$ and a meromorphic function $f(z)$ and $f(z+\omega)$ share (S_1, p), (S_2, m), where $0 \le p < \infty$ and $2 \le m < \infty$, then for $\frac{b^2}{4ac} = \frac{n(n-2)}{(n-1)^2}$,*

$$\begin{aligned} N(r,\mathcal{H}) \le\ & \overline{N}(r,0;f| \ge p+1) + \overline{N}(r,\delta^n_{a,b};f| \ge p+1) + \overline{N}_*(r,1;\mathcal{F},\mathcal{G}) \\ & + \chi_n\Big\{\overline{N}(r,0;f| \le p) + \overline{N}(r,\delta^n_{a,b};f| \le p)\Big\} + \overline{N}_*(r,\infty;f,f(z+\omega)) \\ & + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;f'(z+\omega)), \end{aligned}$$

where $\overline{N}_0(r,0;f'(z))$ is the reduced counting function for the points $\{z \in \mathbb{C} : f'(z) = 0, f(z) \ne 0, \delta^n_{a,b};\ \mathcal{F}(z) \ne 1\}$ and $\overline{N}_0(r,0;f'(z+\omega))$ is defined similarly.

Proof. Since $\mathcal{F} - 1 = \frac{\mathcal{P}(f(z))+d}{-d}$ and $\mathcal{G} - 1 = \frac{\mathcal{P}(f(z+\omega))+d}{-d}$ and $E_f(S_2,m) = E_{f(z+\omega)}(S_2,m)$, we get that $\mathcal{F}$ and $\mathcal{G}$ share $(1,m)$. Next we see that

$$\begin{aligned} \mathcal{F}' &= -\frac{1}{d}\Big\{naf^{n-1}(z) + b(n-1)f^{n-2}(z) + c(n-2)f^{n-3}(z)\Big\}f'(z) \\ &= -\frac{na}{d}f^{n-3}(z)\left(f(z) - \delta^n_{a,b}\right)^2 f'(z), \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}'' =\ & -\frac{na}{d}\Big\{\Big[(n-1)f^2(z) + \frac{b}{na}(n-1)(n-2)f(z) + \frac{c}{na}(n-2)(n-3)\Big]f^{n-4}(z)\left(f'(z)\right)^2 \\ & + f^{n-3}(z)\left(f - \delta^n_{a,b}\right)^2 f''(z)\Big\}. \end{aligned}$$

It is enough to show that

$$\frac{\mathcal{F}''}{\mathcal{F}'} = \frac{2f'(z)}{f(z) - \delta^n_{a,b}} + (n-3)\frac{f'(z)}{f(z)} + \frac{f''(z)}{f''(z)}.$$

Now we have

$$\begin{aligned} \frac{\mathcal{F}''}{\mathcal{F}'} &= \frac{\Big\{(n-1)f^2(z) + \frac{b}{na}(n-1)(n-2)f(z) + \frac{c}{na}(n-2)(n-3)\Big\}}{f(z)(f(z) - \delta^n_{a,b})^2}f'(z) + \frac{f''(z)}{f'(z)} \\ &= \frac{\Big\{n(f(z) - \delta^n_{a,b})^2 - (f(z) - \delta^n_{a,b})(f(z) - 3\,\delta^n_{a,b})\Big\}}{f(z)(f(z) - \delta^n_{a,b})^2}f'(z) + \frac{f''(z)}{f'(z)} \\ &= \frac{\Big\{2f(z) + (n-3)(f(z) - \delta^n_{a,b})\Big\}}{f(z)(f(z) - \delta^n_{a,b})}f'(z) + \frac{f''(z)}{f'(z)} \\ &= \frac{2f'(z)}{f(z) - \delta^n_{a,b}} + (n-3)\frac{f'(z)}{f(z)} + \frac{f''(z)}{f'(z)}. \end{aligned}$$

Similarly, we get

$$\frac{\mathcal{G}''}{\mathcal{G}'} = \frac{2f'(z+\omega)}{f(z+\omega) - \delta^n_{a,b}} + (n-3)\frac{f'(z+\omega)}{f(z+\omega)} + \frac{f''(z+\omega)}{f'(z+\omega)}.$$

Based on the above calculation, we get

$$\begin{aligned} \mathcal{H} =\ & \frac{2f'(z)}{f(z) - \delta^n_{a,b}} - \frac{2f'(z+\omega)}{f(z+\omega) - \delta^n_{a,b}} + \frac{(n-3)f'(z)}{f(z)} - \frac{(n-3)f'(z+\omega)}{f(z+\omega)} \\ & + \frac{f''(z)}{f'(z)} - \frac{f''(z+\omega)}{f'(z+\omega)} - \left(\frac{2\mathcal{F}'}{\mathcal{F} - 1} - \frac{2\mathcal{G}'}{\mathcal{G} - 1}\right). \end{aligned}$$

Since $E_{f(z)}(S_1,0) = E_{f(z+\omega)}(S_1,0)$, so we must have $\overline{N}(r,0;f(z)) + \overline{N}(r,\delta^n_{a,b};f(z)) = \overline{N}(r,0;f(z+\omega)) + \overline{N}(r,\delta^n_{a,b};f(z+\omega))$.

It can also easily be verified that possible poles of $\mathcal{H}$ occur at (i) zeros ($\delta^n_{a,b}$-points) of f and $f(z+\omega)$ when $n \neq 5$ (ii) poles of f and $f(z+\omega)$ with different multiplicities, (iii) 1-points of $\mathcal{F}$ and $\mathcal{G}$ with different multiplicities, (iv) zeros of $f'(z)$ which are not the zeros of $f(z)(f(z)-\delta^n_{a,b})$ and $\mathcal{F}-1$, (v) zeros of $f(z+\omega)$ which are not the zeros of $f(z+\omega)(f(z+\omega)-\delta^n_{a,b})$ and $\mathcal{G}-1$.

Since $\mathcal{H}$ has only simple poles, clearly the lemma follows from the above explanations. □

Lemma 2.3. *Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are as in* (2.1), (2.2) *and let S_i $i=1,2,3$ be defined as in* Theorem 1.2. *If $\mathcal{H} \not\equiv 0$ and a meromorphic function $f(z)$ and $f(z+\omega)$ share (S_1,p), (S_2,m), $(S_3,0)$ where $0 \le p < \infty$ and $2 \le m < \infty$, then for $\frac{b^2}{4ac} = \frac{n(n-2)}{(n-1)^2}$,*

$$\begin{aligned} N(r,\mathcal{H}) \le{} & \overline{N}(r,0;f| \ge p+1) + \overline{N}(r,\delta^n_{a,b};f| \ge p+1) + \overline{N}_*(r,1;\mathcal{F},\mathcal{G}) \\ & + \chi_n \Big\{ \overline{N}(r,0;f| \le p) + \overline{N}(r,\delta^n_{a,b};f| \le p) \Big\} + \overline{N}_*(r,\infty;f,g) \\ & + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g'), \end{aligned}$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function for the points $\{z \in \mathbb{C} : f'(z)=0, f(z) \neq 0,1; \mathcal{F}(z) \neq 1\}$ and $\overline{N}_0(r,0;g')$ is defined similarly.

P r o o f. The proof can be done in the line of proof of Lemma 2.2. □

Lemma 2.4 ([12])**.** *Let f be a non-constant meromorphic function and let*

$$\mathcal{R}(f) = \frac{\sum_{i=1}^{n} a_i f^i}{\sum_{j=1}^{m} b_j f^j},$$

be an irreducible rational function in f with constant coefficients $\{a_i\}$, $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r,\mathcal{R}(f)) = \max\{n,m\}\, T(r,f) + S(r,f).$$

The following lemma can be proved in the line of proof of [2: Lemma 2.10].

Lemma 2.5. *If meromorphic functions $f(z)$ and $f(z+\omega)$ share $(1,m)$, then*

$$\begin{aligned} & \overline{N}(r,1;f) + \overline{N}(r,1;f(z+\omega)) - N(r,1;f \mid= 1) + \Big(m - \frac{1}{2}\Big)\overline{N}_*(r,1;f(z),f(z+\omega)) \\ & \le \frac{1}{2}\left[N(r,1;f(z)) + N(r,1;f(z+\omega))\right]. \end{aligned}$$

Lemma 2.6 ([3: Lemma 2.6])**.** *Let $\Phi(z) = \mathcal{A}(z^{n-m}-1)^2 - \mathcal{B}(z^{n-2m}-1)(z^n-1)$, where $\mathcal{A},\mathcal{B} \in \mathbb{C} \setminus \{0\}$, $\frac{\mathcal{A}}{\mathcal{B}} = \frac{n(n-2m)}{(n-m)^2}$, then $\phi(z)$ has exactly one multiple zero of multiplicity 4, which is 1, i.e.,*

$$\Phi(z) = (z-1)^4 \prod_{k=1}^{2n-2m-4} (z-\beta_k),$$

where $\beta_i \neq \beta_j$, for $i \neq j$, $\beta_k \in \mathbb{C} - \{0,1\}$, for $i,j \in \{1,2,\ldots,2n-2m-4\}$.

Lemma 2.7. *For an integer $n \ge 4$, if meromorphic functions $f(z)$ and $f(z+\omega)$ share $\big(\{0,\delta^n_{a,b}\},0\big)$ and $\mathcal{P}(f(z)) \equiv \mathcal{P}(f(z+\omega))$ then $f(z) \equiv f(z+\omega)$.*

P r o o f. From the given condition we can write

$$f^{n-2}(z)[f(z)-\alpha][f(z)-\beta] \equiv f^{n-2}(z+\omega)[f(z+\omega)-\alpha][f(z+\omega)-\beta]. \tag{2.4}$$

Clearly (2.4) implies f and g share (∞,∞). As

$$E_{f(z)}\big(\{0,\delta^n_{a,b}\},0\big) = E_{f(z+\omega)}\big(\{0,\delta^n_{a,b}\},0\big),$$

it follows that if z_0 is a zero of $f(z)$ $(f(z+\omega))$, then it can not be a $\delta^n_{a,b}$-point of $f(z+\omega)$ $(f(z))$ as none of α and β is zero. So $f(z)$ and $f(z+\omega)$ share $(0,\infty)$ and $\big(\delta^n_{a,b},\infty\big)$.

Suppose $h(z)=\frac{f(z)}{f(z+\omega)}$. Clearly h has no zero and no pole.

Now substituting $f(z)=h(z)f(z+\omega)$ in (2.4) we get

$$a(h^n-1)f^2(z+\omega)+b(h^{n-1}-1)f(z+\omega)+c(h^{n-2}-1)\equiv 0. \tag{2.5}$$

Suppose h is not a constant. Then by a simple calculation, we have from (2.5) with $m=1$, $A=b^2$ and $B=4ac$ in Lemma 2.6,

$$\begin{aligned}&\Big\{a(h^n-1)f(z+\omega)+\frac{b}{2}(h^{n-1}-1)\Big\}^2\\ &\equiv \frac{b^2(h^{n-1}-1)^2-4ac(h^{n-2}-1)(h^n-1)}{4}\\ &=\frac{\phi(h)}{4}=\frac{1}{4}(h-1)^4\prod_{k=1}^{2n-6}(h-\beta_k),\end{aligned} \tag{2.6}$$

where $\beta_k\in\mathbb{C}\setminus\{0,1\}(k=1,2,\ldots,2n-6)$ are distinct.

From (2.6) we see that $h-\beta_k$ $(k=1,2,\ldots,2n-6)$ have multiplicity of order at least 2. So by the *Second Fundamental Theorem*, we get

$$\begin{aligned}(2n-6)T(r,h) &\le \overline{N}(r,\infty;h)+\overline{N}(r,0;h)+\sum_{j=1}^{2n-6}\overline{N}\,(r,\beta_j;h)+S(r,h)\\ &\le \frac{1}{2}\sum_{j=1}^{2n-6}N\,(r,\beta_j;h)+S(r,h)\\ &\le (n-3)T(r,h)+S(r,h),\end{aligned}$$

which is a contradiction for $n\ge 4$. So h is a constant. Again since $f(z+\omega)$ is non-constant, so from (2.5), we have $h^n-1=0$, $h^{n-1}-1=0$ and $h^{n-2}-1=0$. It follows that $h^d-1=0$, where $d=\gcd(n,n-1,n-2)=1$, i.e., $h=1$ and hence $f(z)\equiv f(z+\omega)$. □

Lemma 2.8. *Let* $n\ge 3$ *and* S_i, $i=1,\,2$ *be as in* Theorem 1.1. *Also let meromorphic functions* $f(z)$ *and* $f(z+\omega)$ *share* (S_1,p), (S_2,m), *where* $p<\infty$. *If* $\mathcal{F}$ *and* $\mathcal{G}$ *are given by* (2.1) *and* $\Psi\not\equiv 0$, *then*

$$\begin{aligned}&(3p+2)\Big\{\overline{N}\,(r,0;f(z)\mid\ge p+1)+\overline{N}\,\big(r,\delta^n_{a,b};f(z)\mid\ge p+1\big)\Big\}\\ &\le \overline{N}_*(r,1;\mathcal{F},\mathcal{G})+\overline{N}(r,\infty;f(z))+\overline{N}(r,\infty;f(z+\omega))\\ &\quad +S(r,f)+S(r,f(z+\omega)).\end{aligned}$$

P r o o f. By assumptions $\mathcal{F}$ and $\mathcal{G}$ share $(1,m)$. Also we see that

$$\Psi=\frac{naf^{n-3}(z)\Big(f(z)-\delta^n_{a,b}\Big)^2f'(z)}{-d(\mathcal{F}-1)}-\frac{naf^{n-3}(z+\omega)\Big(f(z+\omega)-\delta^n_{a,b}\Big)^2f'(z+\omega)}{-d(\mathcal{G}-1)}.$$

Let z_0 be a zero or a $\delta^n_{a,b}$-point of $f(z)$ with multiplicity r. Since $E_{f(z)}(S_1,p)=E_{f(z+\omega)}(S_1,p)$ then that would be a zero of Ψ of multiplicity $\min\{(n-3)r+r-1,\ 2r+r-1\}$, i.e., of multiplicity

$\min\{(n-2)r-1,\ 3r-1\} = 3r-1$ if $r \le p$ and a zero of multiplicity at least $\min\{(n-3)(p+1)+p,\ 2(p+1)+p\}$, i.e., a zero of multiplicity at least $\min\{(n-2)p+(n-3),\ 3p+2\} = 3p+2$ if $r > p$.

So by a simple calculation, we can write

$$\begin{aligned}
&(3p+2)\big\{\overline{N}(r,0;f(z)\mid\ge p+1)+\overline{N}\left(r,\delta^n_{a,b};f(z)\mid\ge p+1\right)\big\}\\
&\le N(r,0;\Psi)\\
&\le T(r,\Psi)\\
&\le N(r,\infty;\Psi)+S(r,\mathcal{F})+S(r,\mathcal{G})\\
&\le \overline{N}_*(r,1;\mathcal{F},\mathcal{G})+\overline{N}_*(r,\infty;f(z),f(z+\omega))\\
&\quad + S(r,f(z))+S(r,f(z+\omega))\\
&\le \overline{N}_*(r,1;\mathcal{F},\mathcal{G})+\overline{N}(r,\infty;f(z))+\overline{N}(r,\infty;f(z+\omega))\\
&\quad + S(r,f(z))+S(r,f(z+\omega)). \qquad \square
\end{aligned}$$

Lemma 2.9. *Let $n \ge 3$ and S_i, $i = 1, 2, 3$ be as in* Theorem 1.2. *Also let meromorphic functions $f(z)$ and $f(z+\omega)$ share (S_1,p), (S_2,m), (S_3,k), where $p<\infty$. If $\mathcal{F}$, $\mathcal{G}$ are given by* (2.1) *and $\Psi \not\equiv 0$, then*

$$\begin{aligned}
&(3p+2)\big\{\overline{N}\left(r,0;f(z)\mid\ge p+1\right)\\
&\quad +\overline{N}\left(r,\delta^n_{a,b};f(z)\mid\ge p+1\right)\big\}\\
&\le \overline{N}_*(r,1;\mathcal{F},\mathcal{G})+\overline{N}_*(r,\infty;f(z),f(z+\omega))\\
&\quad + S(r,f(z))+S(r,f(z+\omega)).
\end{aligned}$$

P r o o f. The proof can be carried out in the line of the proof of Lemma 2.8. □

Lemma 2.10. *Let S_i, $i = 1, 2$ be defined as in* Theorem 1.1 *and $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be given by* (2.1) *and* (2.2). *If meromorphic functions $f(z)$ and $f(z+\omega)$ share (S_1,p), (S_2,m), where $0 \le p < \infty$, $2 \le m < \infty$ and $\mathcal{H} \not\equiv 0$, then*

$$\begin{aligned}
&(n+1)\left\{T(r,f(z))+T(r,f(z+\omega))\right\}\\
&\le 2\big\{\overline{N}(r,0;f(z))+\overline{N}\big(r,\delta^n_{a,b};f(z)\big)\big\}+\overline{N}(r,0;f(z)\mid\ge p+1)\\
&\quad +\overline{N}\left(r,\delta^n_{a,b};f(z)\mid\ge p+1\right)+\chi_n\big\{\overline{N}(r,0;f(z)\mid\le p)\\
&\quad +\overline{N}\left(r,\delta^n_{a,b};f(z)\mid\le p\right)\big\}+2\overline{N}(r,\infty;f(z))\\
&\quad +2\overline{N}(r,\infty;f(z+\omega))+\frac{1}{2}\big[N(r,1;\mathcal{F})+N(r,1;\mathcal{G})\big]\\
&\quad -\Big(m-\frac{3}{2}\Big)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\\
&\quad +S(r,f(z))+S(r,f(z+\omega)).
\end{aligned}$$

P r o o f. By applying *Second Fundamental Theorem* and noting that $f(z)$ is of finite order, we get

$$\begin{aligned}
&(n+1)\Big\{T(r,f(z))+T(r,f(z+\omega))\Big\}\\
&\le \overline{N}(r,1;\mathcal{F})+\overline{N}(r,0;f(z))+\overline{N}\left(r,\delta^n_{a,b};f(z)\right)\\
&\quad +\overline{N}(r,\infty;f(z))+\overline{N}(r,1;\mathcal{G})+\overline{N}(r,0;f(z+\omega))\\
&\quad +\overline{N}\left(r,\delta^n_{a,b};f(z+\omega)\right)+\overline{N}(r,\infty;f(z+\omega))-N_0(r,0;f'(z))\\
&\quad -N_0(r,0;f'(z+\omega))+S(r,f(z))+S(r,f(z+\omega)).
\end{aligned} \tag{2.7}$$

Using Lemmas 2.1, 2.2, 2.4 and 2.5, we see that

$$\begin{aligned}
&\overline{N}(r,1;\mathcal{F})+\overline{N}(r,1;\mathcal{G})\\
&\le \frac{1}{2}\big[N(r,1;\mathcal{F})+N(r,1;\mathcal{G})\big]+N(r,1;\mathcal{F}\mid=1)-\Big(m-\frac{1}{2}\Big)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\\
&\le \frac{1}{2}\big[N(r,1;\mathcal{F})+N(r,1;\mathcal{G})\big]+\overline{N}(r,0;f(z)\mid\ge p+1)+\overline{N}\,(r,1;f(z)\mid\ge p+1)\\
&\quad+\chi_n\big\{\overline{N}(r,0;f(z)\mid\le p)+\overline{N}\,\big(r,\delta^n_{a,b};f(z)\mid\le p\big)\,\big\}\\
&\quad+\overline{N}_*(r,\infty;f(z),f(z+\omega))-\Big(m-\frac{3}{2}\Big)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\\
&\quad+\overline{N}_0(r,0;f'(z))+\overline{N}_0(r,0;f'(z+\omega))\\
&\quad+S(r,f(z))+S(r,f(z+\omega)).
\end{aligned}\tag{2.8}$$

Using (2.8) in (2.7) and noting that

$$\overline{N}(r,0;f(z))+\overline{N}\,\big(r,\delta^n_{a,b};f(z)\big)=\overline{N}(r,0;f(z+\omega))+\overline{N}\,\big(r,\delta^n_{a,b};f(z+\omega)\big)\,,$$

the lemma follows. □

Lemma 2.11. *Let S_i, $i=1,\ 2,\ 3$ be defined as in* Theorem 1.2 *and $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be given by* (2.1) *and* (2.2)*). If meromorphic functions $f(z)$ and $f(z+\omega)$ share (S_1,p), (S_2,m) and (S_3,k), where $0\le p<\infty$, $2\le m<\infty$ and $\mathcal{H}\not\equiv 0$, then*

$$\begin{aligned}
&(n+1)\,\big\{T(r,f(z))+T(r,f(z+\omega))\big\}\\
&\le 2\big\{\overline{N}(r,0;f(z))+\overline{N}\,\big(r,\delta^n_{a,b};f(z)\big)\,\big\}+\overline{N}(r,0;f(z)\mid\ge p+1)\\
&\quad+\overline{N}\,\big(r,\delta^n_{a,b};f(z)\mid\ge p+1\big)+\chi_n\big\{\overline{N}(r,0;f(z)\mid\le p)\\
&\quad+\overline{N}(r,\infty;f(z))+\overline{N}(r,\infty;f(z+\omega))\\
&\quad+\overline{N}_*(r,\infty;f(z),f(z+\omega))+\frac{1}{2}\big[N(r,1;\mathcal{F})+N(r,1;\mathcal{G})\big]\\
&\quad-\Big(m-\frac{3}{2}\Big)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+S(r,f)+S(r,g).
\end{aligned}$$

P r o o f. The proof can carried out in the lie of the proof of Lemma 2.10 □

Lemma 2.12. *Let S_1 be defined as in* Theorem 1.1 *with $\frac{a^2}{4b}=\frac{n(n-2m)}{(n-m)^2}$ and $\mathcal{F}$, $\mathcal{G}$ be given by* (2.1) *where $n\ge 3$ and they share $(1,m)$ for $2\le m\le\infty$. Then*

$$\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\le\frac{1}{2m-1}\big[\overline{N}(r,\infty;f(z))+\overline{N}\,(r,\infty;f(z+\omega))\,\big]+S(r,f(z))+S(r,f(z+\omega)).$$

P r o o f. By Lemma 2.8 with $p=0$, we have

$$\begin{aligned}
\overline{N}_*(r,1;\mathcal{F},\mathcal{G})&\le\frac{1}{m}\big[\overline{N}(r,0;f(z))+\overline{N}\,\big(r,\delta^n_{a,b};f(z)\big)\,\big]+S(r,f(z))\\
&\le\frac{1}{2m}\big[\overline{N}(r,\infty;f(z))+\overline{N}\,\big(r,\infty;f(z+\omega))+\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\big)\,\big]\\
&\quad+S(r,f(z))+S(r,f(z+\omega)),
\end{aligned}$$

i.e.,

$$\begin{aligned}
\overline{N}_*(r,1;\mathcal{F},\mathcal{G})&\le\frac{1}{2m-1}\big[\overline{N}(r,\infty;f(z))+\overline{N}\,(r,\infty;f(z+\omega))\,\big]+S(r,f(z))\\
&\quad+S(r,f(z+\omega)).
\end{aligned}\tag{2.9}$$

□

Lemma 2.13 ([14: Lemma 6]). *If $\mathcal{H} \equiv 0$, then $\mathcal{F}$ and $\mathcal{G}$ share $(1,\infty)$. If further $\mathcal{F}$ and $\mathcal{G}$ share $(\infty,0)$, then $\mathcal{F}$, $\mathcal{G}$ share (∞,∞).*

Lemma 2.14. *Let $\mathcal{F}$ and $\mathcal{G}$ be given by (2.1) and they share $(1,m)$. Let $\alpha_1,\alpha_2,\dots,\alpha_n$ be the distinct elements of the set $\{z : az^n + bz^{n-1} + cz^{n-2} + d = 0\}$, where $d \in \mathbb{C} \setminus \{0,\ \sigma,\ \frac{\sigma}{2}\}$ and $n \geq 3$. Then*

$$\overline{N}_L(r,1;\mathcal{F}) \leq \frac{1}{m+1}\left[\overline{N}(r,0;f(z)) + \overline{N}(r,\infty;f(z)) - N_{\otimes}(r,0;f'(z))\right] + S(r,f(z)),$$

where $N_{\otimes}(r,0;f'(z))$ is the counting function of those 0-points of $f'(z)$ which are not in $f^{-1}(\{0,\alpha_1,\alpha_2,\dots,\alpha_n\})$.

Proof. The proof can be carried out along the lines of the proof of [1: Lemma 2.14]. □

3. Proofs of the theorems

Proof of Theorem 1.1. Part (i). Let f and $f(z+\omega)$ be non-constant meromorphic function such that $E_f(S_1,1) = E_{f(z+\omega)}(S_1,1)$ and $E_f(S_2,3) = E_{f(z+\omega)}(S_2,3)$. Suppose $\mathcal{F}$ and $\mathcal{G}$ be given by (2.1). Then $\mathcal{F}$ and $\mathcal{G}$ share $(1,3)$. We consider the following cases.
Case 1. Suppose that $\Psi \not\equiv 0$.
Subcase 1.1. Let $\mathcal{H} \not\equiv 0$. Then for $n = 5$ using Lemma 2.10 for $p = 1$, $m = 3$, Lemma 2.8 for $p = 1$, $p = 0$, Lemma 2.4 and Lemma 2.12 for $m = 3$, we obtain

$$\begin{aligned}
&(n+1)\left\{T(r,f(z)) + T(r,f(z+\omega))\right\}\\
&\leq 2\left\{\overline{N}(r,0;f(z)) + \overline{N}\left(r,\delta^n_{a,b};f\right)\right\} + \frac{1}{5}\left\{\overline{N}(r,\infty;f) + \overline{N}(r,\infty;f(z+\omega))\right.\\
&\quad \left. + \overline{N}_*(r,1;\mathcal{F},\mathcal{G})\right\}\\
&\quad + 2\overline{N}(r,\infty;f(z)) + 2\overline{N}(r,\infty;f(z+\omega)) + \frac{1}{2}\Big[N(r,1;\mathcal{F})\\
&\quad + N(r,1;\mathcal{G})\Big] - \frac{3}{2}\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\\
&\quad + S(r,f(z)) + S(r,f(z+\omega))\\
&\leq \frac{16}{5}\left\{\overline{N}(r,\infty;f(z)) + \overline{N}(r,\infty;f(z+\omega))\right\} + \frac{n}{2}\left[T(r,f(z)) + T(r,f(z+\omega))\right]\\
&\quad - \frac{3}{10}\overline{N}_*(r,1;\mathcal{F},\mathcal{G}) + S(r,f(z)) + S(r,f(z+\omega))\\
&\leq \left\{\frac{n}{2} + \frac{16}{5}\right\}\left[T(r,f(z)) + T(r,f(z+\omega))\right] + S(r,f(z)) + S(r,f(z+\omega))\\
&\leq \left\{\frac{n}{2} + \frac{16}{5}\right\}\left[T(r,f(z)) + T(r,f(z+\omega))\right] + S(r,f(z)) + S(r,f(z+\omega))
\end{aligned}$$

which contradicts $n = 5$.

For $n \geq 6$, in a similar way as above we get

$$\begin{aligned}
&(n+1)\left\{T(r,f(z)) + T(r,f(z+\omega))\right\}\\
&\leq \left\{\frac{n}{2} + \frac{7}{2}\right\}\left[T(r,f(z)) + T(r,f(z+\omega))\right] + S(r,f(z)) + S(r,f(z+\omega)),
\end{aligned}$$

which is again a contradiction for $n \geq 6$.
The rest of the proof can be carried out in the line of proof of part (i) of this theorem.

Subcase 1.2. Let $\mathcal{H} \equiv 0$. Then from (2.2) we get

$$\frac{1}{\mathcal{F}-1} \equiv \frac{\mathcal{A}}{\mathcal{G}-1} + \mathcal{B}, \tag{3.1}$$

where $\mathcal{A}(\neq 0)$ and $\mathcal{B}$ are two constants. So in view of Lemma 2.4, from (3.1) we get

$$T(r, f(z)) = T(r, f(z+\omega)) + O(1). \tag{3.2}$$

Subcase 1.2.1. Suppose $\mathcal{B} \neq 0$, then from (3.1) we get

$$\mathcal{F} - 1 \equiv \frac{\mathcal{G}-1}{\mathcal{B}\mathcal{G} + \mathcal{A} - \mathcal{B}}. \tag{3.3}$$

Subcase 1.2.1.1. If $\mathcal{A} - \mathcal{B} \neq 0$, then noting that $\frac{\mathcal{B}-\mathcal{A}}{\mathcal{B}} \neq 1$, from (3.3) we get

$$\overline{N}\Big(r, \frac{\mathcal{B}-\mathcal{A}}{\mathcal{B}}; \mathcal{G}\Big) = \overline{N}(r, \infty; \mathcal{F}).$$

Now let us consider the following subcases.
Subcase 1.2.1.1.1. Suppose that $\frac{\mathcal{B}-\mathcal{A}}{\mathcal{B}} \neq \frac{\sigma}{d}$.

Therefore in view of equation (3.2) using the *Second Fundamental Theorem* and noting that $f(z)$ is of finite order, we have

$$\begin{aligned}(n+1)T(r, f(z+\omega)) &\leq \overline{N}(r, 0; f(z+\omega)) + \overline{N}(r, \delta^n_{a,b}; f(z+\omega)) + \overline{N}(r, \infty; f(z+\omega)) \\ &\quad + \overline{N}\left(r, \frac{\mathcal{B}-\mathcal{A}}{\mathcal{B}}; \mathcal{G}\right) + S(r, f(z+\omega)) \\ &\leq 3\, T(r, f(z+\omega)) + \overline{N}(r, \infty; f(z)) + S(r, f(z+\omega)) \\ &\leq 4\, T(r, f(z+\omega)) + S(r, f(z)),\end{aligned}$$

which contradicts $n \geq 4$.
Subcase 1.2.1.1.2. Suppose that $\frac{\mathcal{B}-\mathcal{A}}{\mathcal{B}} = \frac{\sigma}{d}$. Since $\frac{b^2}{4ac} = \frac{n(n-2)}{(n-1)^2}$, then from Lemma 2.2, we know that

$$\mathcal{G}' = -\frac{na}{d} f^{n-3}(z+\omega)\left(f(z+\omega) - \delta^n_{a,b}\right)^2 f'(z+\omega). \tag{3.4}$$

We note that $\sigma \neq 0$, otherwise n will be purely imaginary, it follows that $\mathcal{P}(z) + d$ is critically injective. Since any critically injective polynomial can have at most one multiple zero, we have

$$af^n(z+\omega) + bf^{n-1}(z+\omega) + cf^{n-2}(z+\omega) + \sigma = \left(f(z+\omega) - \delta^n_{a,b}\right)^3 \prod_{j=1}^{n-3} (f(z+\omega) - \eta_j),$$

where η_j's are $(n-3)$ distinct zeros of $az^n + bz^{n-1} + cz^{n-2} + \sigma$ such that $\eta_j \neq \delta^n_{a,b}$, 0.

Next from (3.3) we have

$$\mathcal{B}(\mathcal{F}-1) \equiv \frac{-d(\mathcal{G}-1)}{(f(z+\omega) - \delta^n_{a,b})^3 \prod\limits_{j=1}^{n-3} (f(z+\omega) - \eta_j)}. \tag{3.5}$$

Since $E_{f(z)}(S_1, 0) = E_{f(z+\omega)}(S_1, 0)$ so $\delta^n_{a,b}$ points of $f(z+\omega)$ are not poles of $\mathcal{F}$ and hence $\delta^n_{a,b}$ is an e.v.P. of $f(z+\omega)$. Furthermore each η_j point of $f(z+\omega)$ of multiplicity p is a pole of $f(z)$ of multiplicity q (say).

Therefore $p = nq \geq n$. So in view of (3.2) and by applying *Second Fundamental Theorem* and noting that $f(z)$ is of finite order, we get

$$\begin{aligned}(n-2)T(r, f(z+\omega)) &\leq \overline{N}(r, 0; f(z+\omega)) + \overline{N}(r, \delta^n_{a,b}; f(z+\omega)) + \overline{N}(r, \infty; f(z+\omega)) \\ &\quad + \sum_{i=1}^{n-3} \overline{N}(r, \eta_j; f(z+\omega)) + S(r, f(z+\omega)) \\ &\leq \Big(2 + \frac{n-3}{n}\Big) T(r, f(z+\omega)) + S(r, f(z+\omega)),\end{aligned}$$

which contradicts $n \geq 5$.

<u>*Subcase 1.2.1.2.*</u> If $\mathcal{A} - \mathcal{B} = 0$ then from (3.3) we have

$$\frac{\mathcal{G}-1}{\mathcal{F}-1} \equiv \mathcal{B}\mathcal{G} = \mathcal{B}\frac{f^{n-2}(z+\omega)\mathcal{Q}(f(z+\omega))}{-d}, \tag{3.6}$$

i.e., 0's of $f(z+\omega)$ and $\mathcal{Q}(f(z+\omega))$ are poles of $\mathcal{F}$. As $b^2 \neq 4ac$, we know that, the zeros α, β of $\mathcal{Q}(z)$ are simple. Now let each α and β-point of $f(z+\omega)$ is of multiplicity p then it is a pole of f of multiplicity q for some $q \geq 1$. Then from (3.6) we get $p = nq$ i.e., $p \geq n$. Similarly as Subcase 1.2.1.1.2, we can prove here that '0' is an e.v.P. of $f(z+\omega)$.

Next using the *Second Fundamental Theorem* and noting that $f(z)$ is of finite order, we get

$$\begin{aligned}T(r, f(z+\omega)) &\leq \overline{N}(r, \alpha; f(z+\omega)) + \overline{N}(r, \beta; f(z+\omega)) + \overline{N}(r, 0; f(z+\omega)) + S(r, f(z+\omega)) \\ &\leq \frac{2}{n} T(r, f(z+\omega)) + S(r, f(z+\omega)),\end{aligned}$$

which contradicts $n \geq 3$.

<u>*Subcase 1.2.2.*</u> Suppose $\mathcal{B} = 0$, then from (3.1), we get that

$$\mathcal{G} - 1 = \mathcal{A}(\mathcal{F} - 1),$$

i.e.,

$$\mathcal{G}' = \mathcal{A}\mathcal{F}',$$

which implies $\Psi \equiv 0$, a contradiction.

<u>*Case 2.*</u> Let $\Psi \equiv 0$. Then on integration we get

$$\mathcal{G} - 1 = \mathcal{A}(\mathcal{F} - 1),$$

i.e.,

$$\begin{aligned}&af^n(z+\omega) + bf^{n-1}(z+\omega) + cf^{n-2}(z+\omega) \\ &\equiv \mathcal{A}\Big(af^n(z) + bf^{n-1}(z) + cf^{n-2}(z) + d\frac{\mathcal{A}-1}{\mathcal{A}}\Big),\end{aligned} \tag{3.7}$$

i.e.,

$$\begin{aligned}&af^n(z+\omega) + bf^{n-1}(z+\omega) + cf^{n-2}(z+\omega) + d(1-\mathcal{A}) \\ &\equiv \mathcal{A}\Big(af^n(z) + bf^{n-1}(z) + cf^{n-2}(z)\Big).\end{aligned} \tag{3.8}$$

<u>*Subcase 2.1.*</u> Let $\mathcal{A} \neq 1$, then as $d \neq 0$, we have $d\frac{(\mathcal{A}-1)}{\mathcal{A}} \neq 0$. Noting that $\sigma \neq 0$, we have the following subcases.

<u>*Subcase 2.1.1.*</u> Suppose $d\frac{(\mathcal{A}-1)}{\mathcal{A}} = \sigma$, then we claim that $d(1-\mathcal{A}) \neq \sigma$. For if $d(1-\mathcal{A}) = \sigma$, i.e., $\mathcal{A} = \frac{d-\sigma}{d}$ and since $d\frac{(\mathcal{A}-1)}{\mathcal{A}} = \sigma$, i.e., $\mathcal{A} = \frac{d}{d-\sigma}$, it follows that $\frac{d-\sigma}{d} = \frac{d}{d-\sigma}$, i.e., $d = \frac{\sigma}{2}$, a contradiction. Thus $az^n + bw^{n-1} + cw^{n-2} + d(1-\mathcal{A}) = 0$ has only simple roots say α_i for $i = 1, 2, \ldots, n$. So from (3.8) we get

$$\prod_{i=1}^{n}(f(z+\omega) - \alpha_i) \equiv \mathcal{A}f^{n-2}(z)(af^2(z) + bf(z) + c). \tag{3.9}$$

Since $E_{f(z)}(S_1, 0) = E_{f(z+\omega)}(S_1, 0)$ so from (3.9) obviously '0' is an e.v.P. of $f(z)$. So $\delta^n_{a,b}$ points and 0 points of $f(z+\omega)$ corresponds to the '0' point of f. Hence using (3.2) and the *Second Fundamental Theorem* and noting that $f(z)$ is of finite order, in view of (3.2) and (3.9) we get

$$\begin{aligned} nT(r, f(z+\omega)) &\leq \sum_{i=1}^{n} \overline{N}(r, \alpha_i; f(z+\omega)) + N(r, 0; f(z+\omega)) \\ &\quad + N(r, \delta^n_{a,b}; f(z+\omega)) + S(r, f(z+\omega)) \\ &\leq 3T(r, f(z)) + S(r, f(z+\omega)), \end{aligned}$$

which is a contradiction for $n \geq 4$.
Subcase 2.1.2. Suppose $d\frac{(\mathcal{A}-1)}{\mathcal{A}} \neq \sigma$. So, $az^n + bz^{n-1} + cz^{n-2} + d\frac{(\mathcal{A}-1)}{\mathcal{A}} = 0$ has only simple roots say α'_i for $i = 1, 2, \dots, n$.

Therefore from (3.7), we have

$$f^{n-2}(z+\omega)(af^2(z+\omega) + bf(z+\omega) + c) \equiv \mathcal{A} \prod_{i=1}^{n} (f - \alpha'_i). \tag{3.10}$$

By the same argument as used in Subcase 2.1.1., we get a contradiction for $n \geq 5$.
Subcase 2.2. Let $\mathcal{A} = 1$ then we get $\mathcal{P}(f(z+\omega)) \equiv \mathcal{P}(f(z))$. Now applying Lemma 2.7, we get $f(z) \equiv f(z+\omega)$.

Part(ii). Let f and $f(z+\omega)$ be non-constant meromorphic function such that $E_f(S_1, 2) = E_{f(z+\omega)}(S_1, 2)$ and $E_f(S_2, 2) = E_{f(z+\omega)}(S_2, 2)$. Suppose $\mathcal{F}$ and $\mathcal{G}$ be given by (2.1). Then $\mathcal{F}$ and $\mathcal{G}$ share $(1, 2)$. We consider the following cases.
Case 1. Suppose that $\Psi \not\equiv 0$.
Subcase 1.1. Let $\mathcal{H} \not\equiv 0$. Then for $n = 5$ using Lemma 2.10 for $p = 2$, $m = 2$, Lemma 2.8 for $p = 2$, $p = 0$, Lemma 2.4 and Lemma 2.12 for $m = 2$, we obtain

$$\begin{aligned} &(n+1)\left\{T(r, f(z)) + T(r, f(z+\omega))\right\} \\ &\leq 2\left\{\overline{N}(r, 0; f(z)) + \overline{N}\left(r, \delta^n_{a,b}; f\right)\right\} + \frac{1}{8}\left\{\overline{N}(r, \infty; f)\right. \\ &\quad \left. + \overline{N}(r, \infty; f(z+\omega)) + \overline{N}_*(r, 1; \mathcal{F}, \mathcal{G})\right\} \\ &\quad + 2\overline{N}(r, \infty; f(z)) + 2\overline{N}(r, \infty; f(z+\omega)) \\ &\quad + \frac{1}{2}\left[N(r, 1; \mathcal{F}) + N(r, 1; \mathcal{G})\right] - \frac{1}{2}\overline{N}_*(r, 1; \mathcal{F}, \mathcal{G}) \\ &\quad + S(r, f(z)) + S(r, f(z+\omega)) \\ &\leq \frac{25}{8}\left\{\overline{N}(r, \infty; f(z)) + \overline{N}(r, \infty; f(z+\omega))\right\} + \frac{n}{2}\left[T(r, f(z)) + T(r, f(z+\omega))\right] \\ &\quad + \frac{5}{8}\overline{N}_*(r, 1; \mathcal{F}, \mathcal{G}) + S(r, f(z)) + S(r, f(z+\omega)) \\ &\leq \left\{\frac{n}{2} + \frac{25}{8}\right\}\left[T(r, f(z)) + T(r, f(z+\omega))\right] + \frac{5}{24}\left\{\overline{N}(r, \infty; f(z)) + \overline{N}(r, \infty; f(z+\omega))\right\} \\ &\quad + S(r, f(z)) + S(r, f(z+\omega)) \\ &\leq \left\{\frac{n}{2} + \frac{10}{3}\right\}\left[T(r, f(z)) + T(r, f(z+\omega))\right] + S(r, f(z)) + S(r, f(z+\omega)), \end{aligned}$$

which is a contradiction.

For $n \geq 6$, in a similar way as above, we get

$$\begin{aligned}&(n+1)\left\{T(r,f(z))+T(r,f(z+\omega))\right\}\\&\leq \left\{\frac{n}{2}+\frac{7}{2}+\frac{1}{3}\right\}\big[T(r,f(z))\\&\quad+T(r,f(z+\omega))\big]+S(r,f(z))+S(r,f(z+\omega)),\end{aligned}$$

which is again a contradiction since $n \geq 6$.

The rest of the proof can be dealt same as in the line of proof of part (i) of this theorem. □

Proof of Theorem 1.2. The proof can be carried out exactly in the same line of the proof of Theorem 1.1 using Lemmas 2.3, 2.9 and 2.11. So we omit the details. □

4. Some relevant discussions and examples

We are first going to prove the following proposition.

Proposition 4.1. *Any two non-constant meromorphic functions f and g satisfying*

$$f+g=-\frac{2b}{5a} \tag{4.1}$$

share the set $S = \left\{z : az^5 + bz^4 + \frac{4b^2}{15a}z^3 + \frac{8b^5}{9375a^4} = 0\right\}$. That is to show that two distinct meromorphic functions f and g satisfying Corollary 1.1 *when $d = \frac{8b^5}{9375a^4}$.*

Proof. Suppose $f+g=-\frac{2b}{5a}$ and let $c=\frac{4b^2}{15a}$. Then with this transformation, we have

$$\begin{aligned}af^5+bf^4+cf^3&\equiv f^3\Big(af^2+bf+c\Big)\\&\equiv-\Big(g+\frac{2b}{5a}\Big)^3\Big\{a\Big(g+\frac{2b}{5a}\Big)^2-b\Big(g+\frac{2b}{5a}\Big)+\frac{4b^2}{15a}\Big\}\\&\equiv-\Big(ag^5+bg^4+cg^3+\frac{16\,b^5}{9375\,a^4}\Big).\end{aligned}$$

i.e.,

$$af^5+bf^4+cf^3+\frac{8\,b^5}{9375\,a^4}\equiv-\Big(ag^5+bg^4+cg^3+\frac{8\,b^5}{9375\,a^4}\Big).$$

This shows that f and g share the set S CM. This completes the proof of the proposition. □

In view of the Proposition 4.1, for $d=\frac{8b^5}{9375a^4}$, we are now going to show, from the following examples, rather to say from the following counter examples that $f(z)$ and $g(z)=f(z+\omega)$ satisfy (4.1) as well as all the conditions of Corollary 1.1, but they are not identical.

Example 1. Let

$$f(z)=\frac{-4b}{5a}\frac{\sin^4\left(\frac{\pi z}{2\omega}\right)}{2-\sin^2\left(\frac{\pi z}{\omega}\right)}.$$

It is clear that $f(z)+f(z+\omega)=-\frac{2b}{5a}$, but $f(z)\not\equiv f(z+\omega)$.

Example 2. Let

$$f(z) = \frac{-b\, P(e^z) + 5a\sin^2\left(\frac{\pi z}{\omega}\right)}{5aP(e^z)},$$

where $P(z) = \sum_{i=1}^{n} a_{2i-1}z^{2i-1}$, $a_i \in \mathbb{C}$ with $a_{2n-1} \neq 0$, where $e^{\omega} = -1$. It is clear that $f(z) + f(z+\omega) = -\frac{2b}{5a}$, but $f(z) \not\equiv f(z+\omega)$.

The following example shows that the function f considered in Corollary 1.1 could not be of infinite order when $d = \frac{8b^5}{9375a^4}$.

Example 3. Let

$$f(z) = \frac{-\frac{2b}{5a}}{1 + e^{e^z}}.$$

We choose the constant c such that $e^{\omega} = -1$. It is clear that $f(z) + f(z+\omega) = -\frac{2b}{5a}$ and satisfying all the conditions of Corollary 1.1 but $f(z) \not\equiv f(z+\omega)$.

5. An open question

In this paper, we have been able to reduce the lower bound of the cardinality of the main range set to a large extent though we are not sure whether this is the best possible result. So the following question is inevitable.

Question 5.1. What is the best possible cardinality of two set sharing problem for the uniqueness of a meromorphic function and its shift operator?

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