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A Note On Cubic Theta Functions and Their Representations

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Abstract : In this paper we shall discuss various representations of cubic theta functions.

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1. Introduction : Ramanujan defined his theta function

$$f(a,b)=\sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2} \tag{1.1}$$

for $|ab|<1$.
Now ,with the help of famous Jacobi's trple product identity,(1.1) can be put in the form

$$f(a,b)=[-a;ab]_\infty[-b;ab]_\infty[ab;ah]_\infty \tag{1.2}$$

where the symbol

$$[\alpha;\beta]_\infty=\Pi_{r=0}^{\infty}(1-\alpha\beta^r)$$

for α and β real or complex and $|\beta|<1$. The most important cases of $f(a,b)$ are

$$\phi(q)=f(q,q)=\sum_{n=-\infty}^{\infty} q^{n^2}=\frac{[-q;-q]_\infty}{[q;-q]_\infty}=[q^2;q^2]_\infty[-q;q^2]_\infty^2 \tag{1.3}$$

$$\psi(q) = f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{[q^2; q^2]_\infty}{[q; q^2]_\infty} \tag{1.4}$$

and

$$f(-q) = f(-q; q^2) = \sum_{n=-\infty}^{\infty} (-)^n q^{n(3n-1)/2} = [q; q]_\infty \tag{1.5}$$

We also have,

$$\chi(-q) = [q; q^2]_\infty \tag{1.6}$$

Motivated by Jacobi's theta function relation

$$\theta_3^4 = \theta_2^4 + \theta_4^4 \tag{1.7}$$

Borweins [4] discovered its cubic theta functiion analogue

$$a^3(q) = b^3(q) + c^3(q) \tag{1.8}$$

where

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} \tag{1.9}$$

$$b(q) = \sum_{m,n=-\infty}^{\infty} w^{m-n} q^{m^2+mn+n^2} \tag{1.10}$$

and

$$c(q) = \sum_{m,n=-\infty}^{\infty} q^{(m+\frac{1}{3})^2+(m+\frac{1}{3})(n+\frac{1}{3})+(n+\frac{1}{3})^2} \tag{1.11}$$

The functions $a(q)$,$b(q)$ and $c(q)$ are cubic theta functions and θ_2, θ_3 and θ_4 are well known Jacobi's theta functions.

Borweins established the following alternative representations for a(q),b(q) and c(q),

$$a(q) = 1 + 6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right) \tag{1.12}$$

$$a(q) = \phi(q)\phi(q^3) + 4q\varphi(q^2)\varphi(q^6) \tag{1.13}$$

$$b(q) = \frac{1}{2}[3a(q^3) - a(q)] \tag{1.14}$$

$$c(q) = \frac{1}{2}[a(q^{1/3}) - a(q)] \tag{1.15}$$

We shall make use of the follwing known results:

If

$$m = \frac{z_1}{z_3} = \frac{\phi^2(q)}{\phi^2(q^3)}, \tag{1.16}$$

then

$$a(q) = \sqrt{(z_1 z_3)}\frac{m^2+6m-3}{4m} \tag{1.17}$$

$$b(q) = \sqrt{(z_1 z_3)}\frac{(3-m)(9-m^2)}{4m^{2/3}} \tag{1.18}$$

and

$$c(q) = \sqrt{(z_1 z_3)}\frac{3(m+1)(m^2-1)^{1/3}}{4m} \tag{1.19}$$

(cf.Berndt[2;Chapter 33 ,lemma (2.1),p.94]).

Also,we shall use the following known results:

$$a(q) = \sqrt{(z_1 z_3)}\{1 + (\alpha\beta)^{1/4}\} \tag{1.20}$$

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(cf.Berndt[2;p.94])

$$b(q) = \frac{z_1 m^{1/2}\alpha^{1/8}(1-\alpha)^{1/2}}{2^{1/3}\beta^{1/24}(1-\alpha)^{1/6}} \tag{1.21}$$

$$c(q) = \frac{3z_1\beta^{1/8}(1-\beta)^{1/2}}{2^{1/3}m^{3/2}\alpha^{1/24}(1-\alpha)^{1/6}} \tag{1.22}$$

(cf.Berndt[2;equations (3.16) and (3.17)p.104)])

where β is of degree 3 over α and m is the multiplier associated with α and β.

2. Evaluation of cubic theta functions and their different representations

We have , if

$$a = \frac{\pi^{1/4}}{\Gamma(3/4)} \quad then \quad \phi(e^{-\pi}) = a \tag{2.1}$$

(cf.Berndt[2;entry1(i),p.325)]

Also, we have

$$\sqrt{z} = \phi(q) \tag{2.2}$$

and

$$\frac{\phi(q)}{\phi(q^3)} = \sqrt{(\frac{z_1}{z_3})} = \sqrt{m} \tag{2.3}$$

(cf.Ramanujan[6;Chapter 17,entry 10(i)])

We also have

$$\frac{\phi(e^{-\pi})}{\phi(e^{-3\pi})} = (6\sqrt{3}-9)^{1/4} \tag{2.4}$$

(Berndt[2;Chapter 35,entry 3 ,p.327])

Now,for $q = e^{-\pi}$,(2.2) yields

$$\sqrt{m} = \sqrt{(\frac{z_1}{z_3})} = \frac{\phi(e^{-\pi})}{\Phi(e^{-3\pi})} = (6\sqrt{3}-9)^{1/4} \tag{2.5}$$

Thus, we have

$$\sqrt{z_1} = \phi(e^{-\pi}) = a \tag{2.6}$$

and from (2.2)

$$\sqrt{z_3} = \phi(e^{-3\pi}) = \frac{a}{(6\sqrt{3}-9)^{1/4}} \tag{2.7}$$

Now, using (2.5),(2.6) and (2.7) in (1.10)-(1.12), we get the following evaluation of $a(q)$, $b(q)$ and $c(q)$ for $q = e^{-\pi}$, after some simplification,

$$a(e^{-\pi}) = \frac{a^2}{(6\sqrt{3}-9)^{1/4}} \frac{3^{1/4}}{2\sqrt{2}} \{3(2\sqrt{3})^{1/2} - \sqrt{3} + 1\} \tag{2.8}$$

$$b(e^{-\pi}) = \frac{a^2(3 - \sqrt{(6\sqrt{3}-9)})}{2(6\sqrt{3}-9)^{1/4}\{2(\sqrt{3}-1)\}^{1/3}} \tag{2.9}$$

$$c(e^{-\pi}) = \frac{a^2(6\sqrt{3}-9)^{1/4}}{(6-2\sqrt{3})}(1+\sqrt{(6\sqrt{3}-9)}) \tag{2.10}$$

Now, using the entries 10,11,12 of Chapter 17 of Ramanujan [6],we get

$$\phi(q)\phi(q^3) = \sqrt{(z_1 z_3)} \tag{2.11}$$

$$4q\psi^2(q)\psi^2(q^3) = z_1 z_3(\alpha\beta)^{1/4} \tag{2.12}$$

Thus, from (1.20),we have,

$$a(q) = \frac{\phi^2(q)\phi^2(q^3) + 4q\psi^2(q)\psi^2(q^3)}{\phi(q)\phi(q^3)} \tag{2.13}$$

Also, we have

$$4q\psi^2(q^2)\psi(q^6) = \sqrt{(z_1 z_3)}(\alpha\beta)^{1/4} \tag{2.14}$$

Now, (1.20),(2.11) and (2.14)yield

$$a(q) = \phi(q)\phi(q^3) + 4q\psi(q^2)\psi(q^6) \tag{2.15}$$

which is the known result (1.13).
Again,

$$4q\frac{\psi^2(-q)\psi^2(-q^3)}{\phi^2(-q^2)\phi^2(-q^6)} = (\alpha\beta)^{\frac{1}{4}} \tag{2.16}$$

Now, from (1.20),(2.11)and (2.16), we get

$$a(q)=\phi(q)\phi(q^3)\Big\{1+4q\frac{\psi^2(-q)\psi^2(-q^3)}{\phi^2(-q^2)\phi^2(-q^6)}\Big\} \tag{2.17}$$

Again, we have

$$4q\frac{f^6(q)f^6(q^3)}{\phi(-q)\phi(-q^3)\phi^5(q)\phi^5(q^3)}=(\alpha\beta)^{\frac{1}{4}} \tag{2.18}$$

Hence, from (1.20),(2.11) and (2.18), we get

$$a(q)=\phi(q)\phi(q^3)+4q\frac{f^6(q)f^6(q^3)}{\phi(-q)\phi(-q^3)\phi^4(q)\phi^4(q^3)} \tag{2.19}$$

Also , one can easily get

$$4q\frac{f6(-q)f^6(-q^3)}{\phi^4(-q)\phi^4(-q^3)\phi^2(q)\phi^2(q^3)}=(\alpha\beta)^{\frac{1}{4}} \tag{2.20}$$

Thus, we have

$$a(q)=\phi(q)\phi(q^3)+\frac{4qf^6(-q)f^6(-q^3)}{\phi^4(-q)\phi^4(-q^3)\phi(q)\phi(q^3)} \tag{2.21}$$

We also have

$$\frac{4qf^3(-q)f^3(-q^6)}{\phi^2(-q^2)\phi^2(-q^6)\phi(q)\phi(q^3)}=(\alpha\beta)^{\frac{1}{4}} \tag{2.22}$$

Thus, using (1.20), (2.11) and (2.22), we get

$$a(q)=\phi(q)\phi(q^3)+\frac{4qf^3(-q)f^3(-q^6)}{\phi^2(-q^2)\phi^2(-q^6)} \tag{2.23}$$

We, again, have

$$4q\Big\{\frac{f^3(-q^4)f^3(-q^{12})}{\phi(-q^2)\phi(-q^6)}\Big\}^{1/2}=\surd(z_1z_3)(\alpha\beta)^{\frac{1}{4}}. \tag{2.24}$$

Thus, from (1.20),(2.11) and (2.24), we get,

$$a(q)=\phi(q)\phi^3(q)+4q\frac{\{f^3(-q^4)f^3(-q^{12})\}^{\frac{1}{2}}}{\{\phi(-q^2)\phi(-q^6)\}^{\frac{1}{2}}} \tag{2.25}$$

Also, we have,

$$\frac{4q\phi^2(q)\phi^2(q^3)}{\chi^6(q)\chi^6(q^3)\phi^2(-q^2)\phi^2(-q^6)} = (\alpha\beta)^{\frac{1}{4}} \tag{2.26}$$

Hence, using (1.20),(2.11) and (2.26), we have

$$a(q) = \phi(q)\phi(q^3)\Big\{1+\frac{4q\phi^2(q)\phi^2(q^3)}{\chi^6(q)\chi^6(q^3)\phi^2(-q^2)\phi^2(-q^6)}\Big\} \tag{2.27}$$

Further,we have

$$\frac{4q\phi^2(-q)\phi^2(-q^3)}{\phi(q)\phi(q^3)\chi^6(-q)\chi^6(-q^3)} = (z_1z_3)^{1/2}(\alpha\beta)^{\frac{1}{4}} \tag{2.28}$$

which ,with the help of (1.20),(2.11) and (2.28) yields

$$a(q) = \phi(q)\phi(q^3) + \frac{4q\phi^2(-q)\phi^2(-q^3)}{\phi(q)\phi(q^3)\chi(-q)\chi^6(-q^3)} \tag{2.29}$$

Again,we have

$$\frac{4q\phi(-q^2)\phi(-q^6)}{\chi^3(-q^2)\chi^3(-q^6)} = (z_1z_3)^{1/2}(\alpha\beta)^{\frac{1}{4}} \tag{2.30}$$

Now, from (1.20),(2.11) and (2.30), we get

$$a(q) = \phi(q)\phi(q^3)+\frac{4q\phi(-q^2)\phi(-q^6)}{\chi^3(-q^2)\chi^3(-q^6)} \tag{2.31}$$

Also, we can have,

$$\frac{f^3(-q)}{f(-q^3)} = \frac{z_1\sqrt{m}}{2^{\frac{1}{3}}}\frac{(1-\alpha)^{\frac{1}{2}}\alpha^{\frac{1}{8}}}{(1-\beta)^{\frac{1}{6}}\beta^{\frac{1}{24}}} \tag{2.32}$$

Hence, (1.21)and (2.32) lead

$$b(q) = \frac{f^3(-q)}{f(-q^3)} \tag{2.33}$$

which is a known result (cf. Berndt[2;chapter 33(5.4),p.109]).
Again, we find that

$$3q^{\frac{1}{3}}\frac{f^3(-q^3)}{f(-q)} = \frac{3z_1\beta^{\frac{1}{8}}(1-\beta)^{\frac{1}{2}}}{2^{\frac{1}{3}}m^{\frac{3}{2}}(1-\alpha)^{\frac{1}{6}}\alpha^{\frac{1}{24}}} \tag{2.34}$$

Now, from (1.22)and (2.34), we have

$$c(q) = \frac{3q^{\frac{1}{3}}f^3(-q^3)}{f(-q)} \tag{2.35}$$

which is another known result (cf. Bendt[2;chapter 33(5.5)p.109]).
Further, we can find that

$$4^{1/6}\left\{\frac{\psi^3(q^2)}{\psi(q^6)}\right\}^{\frac{1}{6}}\left\{\frac{\{\phi^{11}(-q)\phi^2(-q^2)}{\phi^3(-q^3)\phi^2(-q^6)}\right\}^{\frac{1}{6}}$$

$$= (z_1/z_3)^{1/2}(\alpha^3/\beta)^{1/24}\{(1-\alpha)^3/(1-\beta)\}^{1/6} \tag{2.36}$$

Now,(1.21)and (2.36)yield

$$b^6(q) = \frac{\psi^3(q^2)\phi^{11}(-q)\phi^2(-q^2)}{\psi(q^6)\phi^3(-q^3)\phi^2(-q^6)} \tag{2.37}$$

Similarly,we have

$$c^6(q) = \frac{3^6q^2\psi^3(q^6)\phi^{11}(-q^3)\phi^2(-q^6)}{\psi(q^2)\phi^3(-q)\phi^2(-q^2)} \tag{2.38}$$

Also, we can get

$$2^{1/3}\left\{\frac{\phi^3(-q^2)}{\phi(-q^6)}\right\}^{\frac{1}{3}}\left\{\frac{\psi^3(q)}{\psi(q^3)}\right\}^{\frac{1}{3}}\left\{\frac{\phi(q^3)}{\phi^3(q)}\right\}^{\frac{1}{3}}$$

$$= (z_1^3/z_3)^{1/2}\Big\{(1-\alpha)^3/(1-\beta)\Big\}^{1/6}(\alpha^3/\beta)^{1/24} \tag{2.39}$$

Now(1.21)and (2.39)lead to

$$b^3(q) = \frac{\phi^{12}(-q^2)\psi^3(q)\phi(q^3)}{\phi^4(-q^6)\psi(q^3)\phi^3(q)} \tag{2.40}$$

Similarly,we get

$$c^3(q) = 3q\frac{\phi^{12}(-q^6)\psi^3(q^3)\phi(q)}{\phi^4(-q^2)\psi(q)\phi^3(q^3)} \tag{2.41}$$

We can easily obtain

$$2^{1/3}\left\{\frac{\phi^3(q)}{\phi(q^3)}\right\}^{1/3}\left\{\frac{\psi^3(-q)\psi(q^3)}{\psi^3(q)\psi(-q^3)}\right\}^{4/3}\left\{\frac{\psi^3(q)}{\psi(q^3)}\right\}^{1/3}$$

$$= \sqrt{\{\frac{z_1^3}{z_3}\}}(\frac{\alpha^3}{\beta})^{\frac{1}{24}}\left\{\frac{(1-\alpha)^3}{1-\beta}\right\}^{\frac{1}{6}} \tag{2.42}$$

Now, with the help of (1.21)and (2.42), we

$$b^3(q) = \frac{\phi^3(q)\psi^{12}(-q)\psi^3(q^3)}{\phi(q^3)\psi^4(-q^3)\psi^9(q)} \tag{2.43}$$

Similarly,

$$c^3(q) = 3q\frac{\phi^3(q^3)\psi^{12}(-q^3)\psi^3(q)}{\phi(q)\psi^4(-q)\psi^9(q^3)} \tag{2.44}$$

It is easy to establish that

$$b(q) = \frac{\phi(q^3)f^3(q)\phi^3(-q^2)}{\phi^3(q)f(q^3)\phi(-q^6)} \tag{2.45}$$

and

$$c(q) = 3q^{\frac{1}{3}}\frac{\phi(q)f^3(q^3)\phi^3(-q^6)}{\phi^3(q^3)f(q)\phi(-q^2)} \tag{2.46}$$

Applying the known results of Rammanujan [6;Chapter17,entris 10,11 and 12], one can establish several other representations for $b(q)$ and $c(q)$.

3. Generalized cubic theta funchons

Hirschhoun, Gavan and Borwein[4]introduced the functions,

$$a(q,z) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} z^{n-m} \ldots \tag{3.1}$$

$$b(q,z) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} w^{n-m} z^m, (w = e^{2\pi i/3}) \cdots \tag{3.2}$$

$$c(q,z) = q^{\frac{1}{3}} \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} q^{m+n} z^{n-m} \tag{3.3}$$

As $z \to 1$,

$$a(q,1) = a(q), b(q,1) = b(q), c(q,1) = c(q) \tag{3.4}$$

Here we want to point out that in the definition of $a(q,z)$ given above the muttiplier $q^{\frac{1}{3}}$ is not in the definintein of $c(q,z)$ given by Hirschhon et.al.[4].They gave a number of identities with the help of Jacobi's triple product identity. Some of these are mentioned below.

$$a(q,z) = \frac{1}{3}(1+z+z^{-1})\Big[1+6\sum_{n=1}^{\infty}\Big(\frac{q^{3n-2}}{1-q^{3n-2}} - \frac{q^{3n-1}}{1-q^{3n-1}}\Big)$$

$$\times\frac{[q;q]_\infty^2[z^3q^3,q^3/z^3;q^3]_\infty}{[q^3;q^3]_\infty^2[zq,q/z;q]_\infty} + \frac{1}{3}(2-z-z^{-1})\frac{[q;q]_\infty^5}{[q^3;q^3]_\infty^3}[zq,q/z;q]_\infty^2 \qquad (3.5(a))$$

$$a(q,z) = (2+z+z^{-1})\frac{[q^2;q^2]_\infty^2[q;q]_\infty}{[-q^3;q^3]_\infty[q^6;q^6]_\infty}[-zq,-q/z;q]_\infty^2$$

$$-(1+z+z^{-1})\frac{[q^2;q^2]_\infty[q^3;q^3]_\infty[z^3q^3,q^3/z^3;q^3]_\infty}{[-q^3;q^3]_\infty^3[zq,q/z;q]_\infty} \qquad (3.5(b))$$

$$b(q,z) = [q;z]_\infty[q^3;q^3]_\infty\frac{[zq,q/z;q]_\infty}{[zq^3,q^3/z;q^3]_\infty} \qquad (3.6)$$

and

$$c(q,z) = \frac{q^{1/3}(1+z+z^{-1})[q;q]_\infty[q^3;q^3]_\infty[z^3q^3,q^3|z^3;q^3]_\infty}{[zq,q/z;q]_\infty} \qquad (3.7)$$

The above representations are helpful in evaluating $a(q,z), b(q,z)$ and $c(q,z)$ for particular values of z. As an illustration,we evaluate these functions for some special values of z.
If we set $z = w(w = e^{2\pi i/3})$, in (3.5(b)),(3.6)and (3.7), we get

$$a(q,w) = \frac{[a^2;a^2]_\infty^2[q;q]_\infty}{[-q^3;q^3]_\infty[q^6;q^6]_\infty}\Pi_{i=1}^{\infty}(1-q^i+q^{2i})^2 \qquad (3.8)$$

$$b(q,w) = [q;q]_\infty[q^3;q^3]_\infty\ \Pi_{i=1}^{\infty}\Big\{\frac{1+q^i+q^{2i}}{1+q^{3i}+q^{6i}}\Big\} \qquad (3.9)$$

and

$$c(q, w) = 0 \tag{3.10}$$

Next if we put $z = -w$ in (3.5(b))-(3.7) we get

$$a(q, -w) = \frac{3[q^2; q^2]^2_\infty [q; q]_\infty}{[-q^3; q^3]_\infty [q^6; q^6]_\infty} \Pi_{i=1}^{\infty} (1 + q^i + q^{2i})^2$$

$$- \frac{2[q^2; q^2]_\infty [q^3; q^3]_\infty}{[-q^3; q^3]_\infty \Pi_{i=1}^{\infty} (1 - q^i + q^{2i})} \tag{3.11}$$

$$b(q, -w) = [q; q]_\infty [q^3; q^3]_\infty \Pi_{i=1}^{\infty} \left\{ \frac{1 - q^i + q^{2i}}{1 - q^{3i} + q^{6i}} \right\} \tag{3.12}$$

and

$$c(q, -w) = 2q^{\frac{1}{3}} \frac{[q; q]_\infty [q^3; q^3]_\infty [-q^3; q^3]^2_\infty}{\Pi_{i=1}^{\infty} (1 - q^i + q^{2i})} \tag{3.13}$$

Furthur, if we take $z = -1$ in (3.5)-(3.7), we get

$$a(q, -1) = \frac{[q^2; q^2]_\infty [q^3; q^3]_\infty}{[-q^3; q^3]_\infty [-q; q]^2_\infty} \tag{3.14}$$

$$b(q, -1) = \frac{[q; q]_\infty [q^3, q^3]_\infty [-q; q]^2_\infty}{[-q^3; q^3]^2_\infty} \tag{3.15}$$

and

$$c(q - 1) = \frac{-q^{\frac{1}{3}} [q; q]_\infty [q^3; q^3]_\infty [-q^3; q^3]^2_\infty}{[-q; q]^2_\infty} \tag{3.16}$$

Lastly,for $z = 1$ in (3.5(a)),(3.6)and (3.7)we get

$$a(q, 1) = a(q) = 1 + 6 \sum_{n=1}^{\infty} \left\{ \frac{q^{3n-2}}{1 - q^{3n-2}} - \frac{q^{3n-1}}{1 - q^{3n-1}} \right\} \tag{3.17}$$

which is a known result (cf.(Equation 1.6))

$$b(q,1) = b(q) = \frac{[q;q]_\infty^3}{[q^3;q^3]_\infty} \tag{3.18}$$

and

$$c(q,1) = c(q) = 3q^{\frac{1}{3}} \frac{[q^3;q^3]_\infty^3}{[q;q]_\infty} \tag{3.19}$$

4. Generalized cubic theta functions and continued fractions

In this section we shall attempt to show certain intertesting results involving cubic theta functions and continued fractions.

If in (3.6) we replace q by q^k and z by q^i,we get, after some simplification,

$$b(q^k,q^i) = \frac{[q^k,q^k]_\infty[q^{3k};q^{3k}]_\infty[q^i,q^{k-i};q^k]_\infty}{[q^i,q^{3k-i};q^{3k}]_\infty} \tag{4.1}$$

Now putting $k=5$ and $i=1$ in(4.1), we get

$$b(q^5,q) = [q^5;q^5]_\infty[q^{15};q^{15}]_\infty \frac{[q,q^4;q^5]_\infty}{[q,q^{14};q^{15}]_\infty} \tag{4.2}$$

Again, taking $k=5$ and $i=2$ in (4.1), we have

$$b(q^5,q^2) = [q^5;q^5]_\infty[q^{15};q^{15}]_\infty \frac{[q^2,q^3;q^5]_\infty}{[q^2,q^{13};q^{15}]_\infty} \tag{4.3}$$

Now,(4.2) and (4.3) yield the foillowing result with the help of a well known result of Ramanujan(cf.Andrews and Berndt[1; Cor.6.2.6 ,p153])

$$\frac{b(q^5,q)}{b(q^5,q^2)} = \frac{[q^2,q^{13};q^{15}]_\infty}{[q,q^{14};q^{15}]_\infty}\left\{\frac{1}{1+}\;\frac{q}{1+}\;\frac{q^2}{1+}\;\frac{q^3}{1+}\ldots\ldots\right\} \tag{4.4}$$

Next, taking $k=6, i=1$, and $k=6, i=3$, respectively, in (4.1) and using the results thus obtained and another known result (cf. Andrews and Berndt[1;6.2.7 p.154]), we get

$$\frac{b(q^6,q)}{b(q^6,q^3)} = \frac{[q^3,q^{15};q^{18}]_\infty}{[q,q^{17};q^{18}]_\infty}\left\{\frac{1}{1+}\ \frac{q+q^2}{1+}\ \frac{q^2+q^4}{1+}\ \frac{q^3+q^6}{1+}\ \ldots\ldots\right\} \quad (4.5)$$

Further,setting $k=8, i=1$ and $k=8, i=3$,respectively,in (4.1) and making use of the results thus obtained and yet another known result (cf.Andrews and Berndt[1;6.2.38,p154]),we get

$$\frac{b(q^8,q)}{b(q^8,q^3)} = \frac{[q^3,q^{21};q^{24}]_\infty}{[q,q^{23};q^{24}]_\infty}\left\{\frac{1}{1+}\ \frac{q+q^2}{1+}\ \frac{q^4}{1+}\ \frac{q^3+q^6}{1+}\ \ldots\ldots\right\} \quad (4.6)$$

Again,putting $k=4, i=2$ and $k=4, i=1$,respectively,in (4.1) and applying a known result (cf.cf.Andrews and Berndt[1;6.2.22,p150]), alongwith the above two relations ,we get the following result,

$$\frac{b(q^4,q)}{b(q^4,q^2)} = \frac{[q^2,q^{10};q^{12}]_\infty}{[q,q^{11};q^{12}]_\infty}\left\{\frac{1}{1+}\ \frac{q}{1+}\ \frac{q+q^2}{1+}\ \frac{q^3}{1+}\ \frac{q^2+q^4}{1+}\ \ldots\ldots\right\} \quad (4.7)$$

Now, we have from(3.6) and (3.7)

$$b(q,z)c(q,z) = q^{1/3}(1+z+z^{-1}))[q;q]_\infty^2[q^3;q^3]_\infty^2\left\{\frac{[z^3q^3,q^3/z^3;q^3]_\infty}{zq^3,q^3/z;q^3]_\infty}\right\} \quad (4.8)$$

Further,(3.7) can be written with q and z replaced by q^k and q^i,respectively,

$$c(q^k,q^i) = q^{k/3}(1+q+q^{-i})[q^k;q^k]_\infty[q^{3k};q^{3k}]_\infty\left\{\frac{1-q^i}{1-q^{3i}}\right\}\frac{[q^{3i},q^{3k-3i};q^{3k}]_\infty}{[q^i,q^{k-i};q^k]_\infty}$$

$$= \frac{[q^k;q^k]_\infty[q^{3k};q^{3k}]_\infty[q^{3i},q^{3k-3i};q^{3k}]_\infty}{q^i[q^i,q^{k-i};q^k]_\infty} \quad (4.9)$$

Now,setting $k=2, i=1$ and also $k=6, i=1$ in (4.9) and applying a known result(cf.Andrews and Berndt[1;6.2.37,p.154]),alongwith the above two resulting relations ,we get

$$c(q^2,q)c(q^6,q)$$

$$= q^{2/3}\frac{[q^2;q^2]_\infty[q^6;q^6]_\infty^2[q^{18};q^{18}]_\infty[q^3,q^{15};q^{18}]_\infty}{[q;q^2]_\infty^2}\left\{1+\ \frac{q+q^2}{1+}\ \frac{q^2+q^4}{1+}\ \frac{q^3+q^6}{1+}\ \ldots\ldots\right\}$$

(4.10)

Lastly,replacing q by q^2 and z by q in (4.8) and making use of the above resulting relations alongwith yet another known result(cf.Andrews and Berndt[1;6.2.37,p.154]), we get the following relation between generalized cubic theta function and continued fraction

$$q^{1/3}b(q^2,q)c(q^2,q)$$
$$= [q^2;q^2]_\infty^2[q^6;q^6]_\infty^2\left\{1+\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+}\cdots\cdots\right\} \quad (4.11)$$

It is evident that many more results of the type investiugated in this section could also be derived easily.

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